10. 10. 2025 – 2. VAJE

PREDSTAVLJIVA ŠTEVILA

$P(b, t, L, U)$

$$x = \pm m \cdot b^e$$

- $b \in \mathbb{N}$: baza
- $e \in \mathbb{Z}$: eksponent
  $L \leq e \leq U$
- m : mantisa
  $m = 0.c_1c_2\ldots c_t$; $0 \leq c_i \leq b-1$

zahtevamo $c_1 \neq 0$, razen za $e = L$
če je $c_1 \neq 0$, je št. normalizirano, sicer pa je denormalizirano.

- $x \notin P$, poiščemo $fl(x) \in P$, tj. najbližje predstavljivo št.
- relativna napaka : $\dfrac{|fl(x) - x|}{|x|} < u$
- osnovna zaokrožitvena napaka : $u = \frac{1}{2} b^{1-t}$

1.
a) Pretvorite $101110_{(2)}$ in $11001.001_{(2)}$ v desetiški sistem

$101110_{(2)} = 0\cdot1 + 1\cdot2 + 1\cdot4 + 1\cdot8 + 0\cdot16 + 1\cdot32 = 46$

$11001.001 = 1 + 0\cdot2 + 0\cdot4 + 1\cdot8 + 1\cdot16 + 0\cdot2^{-1} + 0\cdot2^{-2} + 1\cdot2^{-3}$
$= 25 + \frac{1}{8} = 25,125$

b) Pretvorite $1362_{(10)}$ in $2.71875_{(10)}$ v dvojiški sistem

$1362 = 2^{10} + 2^8 \ldots = 10101010010$

1362 : 2 = 681
16
02

681 : 2 = 340
08
01
ost. 1

340 : 2 = 170
14
00

338 → 256

1362 = 1024

1362
−1024
338

$\mathrm{Int}(2,71875) = 2$

	celi del:
$0,71875_{(10)} \cdot 2 = 1,4375$	1
$0,4375 \cdot 2 = 0,875$	0
$0,875 \cdot 2 = 1,75$	1
$0,75 \cdot 2 = 1,5$	1
$0,5 \cdot 2 = 1,\underline{0}$	1

$$0,71875_{(10)} = 0,10111_{(2)}$$

$$2,71875_{(10)} = 10,10111_{(2)}$$

(2.) Zapiši vsa normalizirana št. iz množice $P(2,3,-1,3)$.
Katera ležijo na intervalu $(0,1)$?
Koliko denormaliziranih št. ima mn. P?

$$\pm \begin{matrix} 0.100 \\ 0.101 \\ 0.110 \\ 0.111 \end{matrix} \cdot 2^e \quad ; \quad e \in \{-1,0,2,3\}$$

interval $(0,1)$: predznak $+$, e je lahko -1 ali 0

$\rightsquigarrow$ 8 števil

denormalizirana: $c_1 = 0$ in $e = L$

$e = -1$, $c_1 = 0$:
$$\pm \begin{matrix} 0,001 \\ 0,010 \\ 0,011 \end{matrix} \cdot 2^{-1} \qquad \text{(6 števil)}$$

$$(0,000 \cdot 2^{-1})$$

(3.) V formatu $P(2,7,-10,10)$ zapišite št. $x = 13.7$.
Izračunajte relativno napako predstavitve št. in jo primerjajte z u za mn. P.

$b = 2$, $t = 7$, $-10 \leq e \leq 10$

$$13,7_{(10)} = 1101.1\overline{0110} = \overbrace{10000_{(2)}}^{2^4} \cdot \overbrace{0,11011\overline{0110}}^{m}$$

$0,7 \cdot 2 = 1,4$
$0,4 \cdot 2 = 0,8$
$0,8 \cdot 2 = 1,6$
$0,6 \cdot 2 = 1,2$
$0,2 \cdot 2 = 0,4$
$0,4 \cdot 2 = 0,8$
$0,8 \cdot 2 = 1,6$

želimo tako obliko za m

$$\underbrace{1101}, \overline{10110} = 0,1\ldots$$

premikanje dec. vejice = množenje z 2 (v dvojiškem)

$$0,\underbrace{1101\,1\,0\,1}_{t=7}\overline{10}\ \cdot\ 2^4$$

če bi bila za mestom, ko odrežemo
⇒ se samo odreže

$$0,1101\,110$$

$$fe(x) = 2^4 \cdot 0,1101110$$
$$= 1101,110_{(2)} = 13,75_{(10)}$$

relativna napaka:

$$\frac{|13,75 - 13,7|}{13,7} = \frac{0,05}{13,7} = 0,00365 < u$$

osnovna zaokrožitvena napaka:

$$u = \frac{1}{2} \cdot 2^{1-7} = 2^{-1} \cdot 2^{-6} = 2^{-7}$$
$$= 0,00781$$

IEEE754 standard:

- enojna natančnost: $P(2, 24, -125, 128)$

$$(-1)^{\sigma}(1+m)\,2^{\tilde{e}-127}$$

$$\left.\begin{array}{l} m: \text{dolžine } 23 \\ \tilde{e}: \text{dolžine } 8 \\ \sigma: \text{dolžine } 1 \end{array}\right\} 32 \text{ bitov}$$

• dvojna natančnost: $P(2, 53, -1021, 1024)$

$$(-1)^{\sigma}(1+m)\cdot 2^{\tilde{e}-1023}$$

$\left.\begin{array}{l} m: \text{dolžine } 52 \\ \tilde{e}: \text{dolžine } 11 \\ \sigma: \text{dolžine } 1 \end{array}\right\}$ 64 bitov

(4.) Zapišite št. $11_{(10)}$ v IEEE754 standardu z enojno natančnostjo in 2,71875 z dvojno natančnostjo.

$$11_{(10)} = 1011_{(2)} = 1,011 \cdot 2^3$$

$m: 0110\ldots0$

$e = \tilde{e} - 127 \qquad e=3$

$\Rightarrow \tilde{e} = 130_{(10)} =$

$= 10000010_{(2)}$

$\sigma = 0$

$$2,71875_{(10)} = 10,10111_{(2)} = 1,010111 \cdot 2^1$$

$m = 0101110\ldots0$

$e = 1 \quad \Rightarrow \quad \tilde{e} = 1 + 1023 = 1024_{(10)}$

$= 10000000000_{(2)}$

$\sigma = 0$

(5.) $x = 0,1_{(10)}$

a) Pokaži, da velja $x = \sum_{i=1}^{\infty}\left(2^{-4i} + 2^{-4i-1}\right) =$

$$= \sum_{i=1}^{\infty}\left(2^{-4i}\right)\left(1+\frac{1}{2}\right) = \frac{3}{2}\sum_{i=1}^{\infty} 2^{-4i} =$$

$$= \frac{3}{2}\sum_{i=1}^{\infty}\left(\frac{1}{16}\right)^i = \frac{3}{2}\,\frac{1/16}{1-\frac{1}{16}} = \frac{3}{2}\cdot\frac{1}{15} =$$

$$= \frac{1}{2\cdot 5} = 0,1$$

b) Zapišite binarni zapis za x s pomočjo a)

$$0,00011\overline{0011}$$

c) zapišite x v IEEE754 z enojno in dvojno natančnostjo

⑥ Računski stroj uporablja dvojiško aritmetiko z mantiso sode dolžine $t \geq 6$. Naj bo
$x = 2^{-1} + 2^{-k} + 2^{-t}$, $y = 2^{-1} + 2^{-k}$, kjer $k = \frac{t}{2} + 1$

Vrednost izraza $x^2 - y^2$ izračunamo kot $(x * x) - (y * y)$.
Pokaži, da je relativna napaka tega izraza "velika" (večja od osnovne zaokrožitvene napake)
$\Rightarrow$ izračun ni direktno stabilen

$$x = 0,\underbrace{100\ldots0}_{\frac{t}{2}}\underbrace{10\ldots01}_{\frac{t}{2}}$$

$$x \cdot x = (2^{-1} + 2^{-k} + 2^{-t}) \cdot (2^{-1} + 2^{-k} + 2^{-t}) =$$
$$= 2^{-2} + 2^{-k-1} + 2^{-t-1} + 2^{-2k} + 2^{-k-t} + 2^{-k-1} + 2^{-t-1} + 2^{-k-t} + 2^{-2t}$$
$$= 2^{-2} + 2^{-k} + 2^{-t} + 2^{-k-t+1} + 2^{-2k} + 2^{-2t} =$$
$$= 2^{-2} + 2^{-\frac{t}{2}-1} + 2^{-t} + 2^{-\frac{t}{2}-t} + 2^{-2(\frac{t}{2}+1)} + 2^{-2t} =$$
$$= 2^{-2} + 2^{-\frac{t}{2}-1} + 2^{-t} \,\Big\|\, + 2^{-\frac{3}{2}t} + 2^{-t-2} + 2^{-2t}$$

konec mantise, zaokroževanje $-t-2$ določa 2

$t = 6$: $2^{-2} + 2^{-4} + 2^{-6} + 2^{-8} + 2^{-9} + 2^{-12} = 0,01\ldots$

$$fl(x^2) = (2^{-1} + 2^{-\frac{t}{2}} + 2^{-t-1} + 2^{-t}) \cdot 2^{-1} = 2^{-2} + 2^{-\frac{t}{2}-1} + 2^{-t} + 2^{-t-1}$$

$\pm m \cdot 2^{e}$

$m = 0,\underbrace{1\ldots}_{t}$

$e = -1$

$$y = 2^{-1} + 2^{-k}$$

$$y \cdot y = (2^{-1} + 2^{-k}) \cdot (2^{-1} + 2^{-k}) = 2^{-2} + 2^{-k-1} + 2^{-k-1} + 2^{-2k} =$$

$$= 2^{-2} + 2^{-k} + 2^{-2k} =$$

$$= 2^{-2} + 2^{-\frac{t}{2}-1} + 2^{-2(\frac{t}{2}+1)} =$$

$$= 2^{-2} + 2^{-\frac{t}{2}-1} + 2^{-t-2}$$

$$y^2 = 0{,}01 \ldots \underset{-\frac{t}{2}-1}{1} \ldots \underset{-t-2}{1} = 0{,}1 \ldots \underset{-\frac{t}{2}}{1} \ldots \underset{-t-1}{1}$$

zaokrožimo na najbližjo sodo

$$0{,}1 \ldots 1 \ldots 0$$

$$fl(y^2) = 2^{-2} + 2^{-k} = 2^{-2} + 2^{-(\frac{t}{2}+1)}$$

$$fl(\, fl(x^2) - fl(y^2)) =$$

$$= fl(\cancel{2^{-2}} + \cancel{2^{-\frac{t}{2}-1}} + 2^{-t} + 2^{-t-1} - \cancel{2^{-2}} - \cancel{2^{-\frac{t}{2}-1}}) =$$

$$= fl(2^{-t} + 2^{-t-1}) = 2^{-t} + 2^{-t-1}$$

$$0{,}00 \ldots 011 \rightarrow 0{,}11 \cdot 2^{\square}$$

ne zahteva dodatnega zaokroževanja

$$z = x^2 - y^2$$

$$\frac{|fl(z) - z|}{|z|} =$$

$$z = x^2 - y^2 = \cancel{2^{-2}} + \cancel{2^{-\frac{t}{2}-1}} + 2^{-t} + 2^{-\frac{3}{2}t} + \cancel{2^{-t-2}} + 2^{-2t} - \cancel{2^{-2}} - \cancel{2^{-\frac{t}{2}-1}} - \cancel{2^{-t-2}} =$$

$$= 2^{-t} + 2^{-\frac{3}{2}t} + 2^{-2t}$$

$$= \frac{|2^{-t} + 2^{-t-1} - 2^{-t} - 2^{-\frac{3}{2}t} - 2^{-2t}|}{|2^{-t} + 2^{-\frac{3}{2}t} + 2^{-2t}|} =$$

$$= \frac{2^{-t-1} - 2^{-\frac{3}{2}t} - 2^{-2t}}{2^{-t} + 2^{-\frac{3}{2}t} + 2^{-2t}} > 0$$

30.10.2025

NAVADNA ITERACIJA

iskanje ničle fun. f $\longrightarrow$ iskanje negibnih točk fun. g

$$f(x)=0 \iff g(x)=x$$

iteracija: g, x_0 : $x_{r+1}=g(x_r)$

- naj bo α negibna točka:

  $|g'(\alpha)|<1$: α privlačna

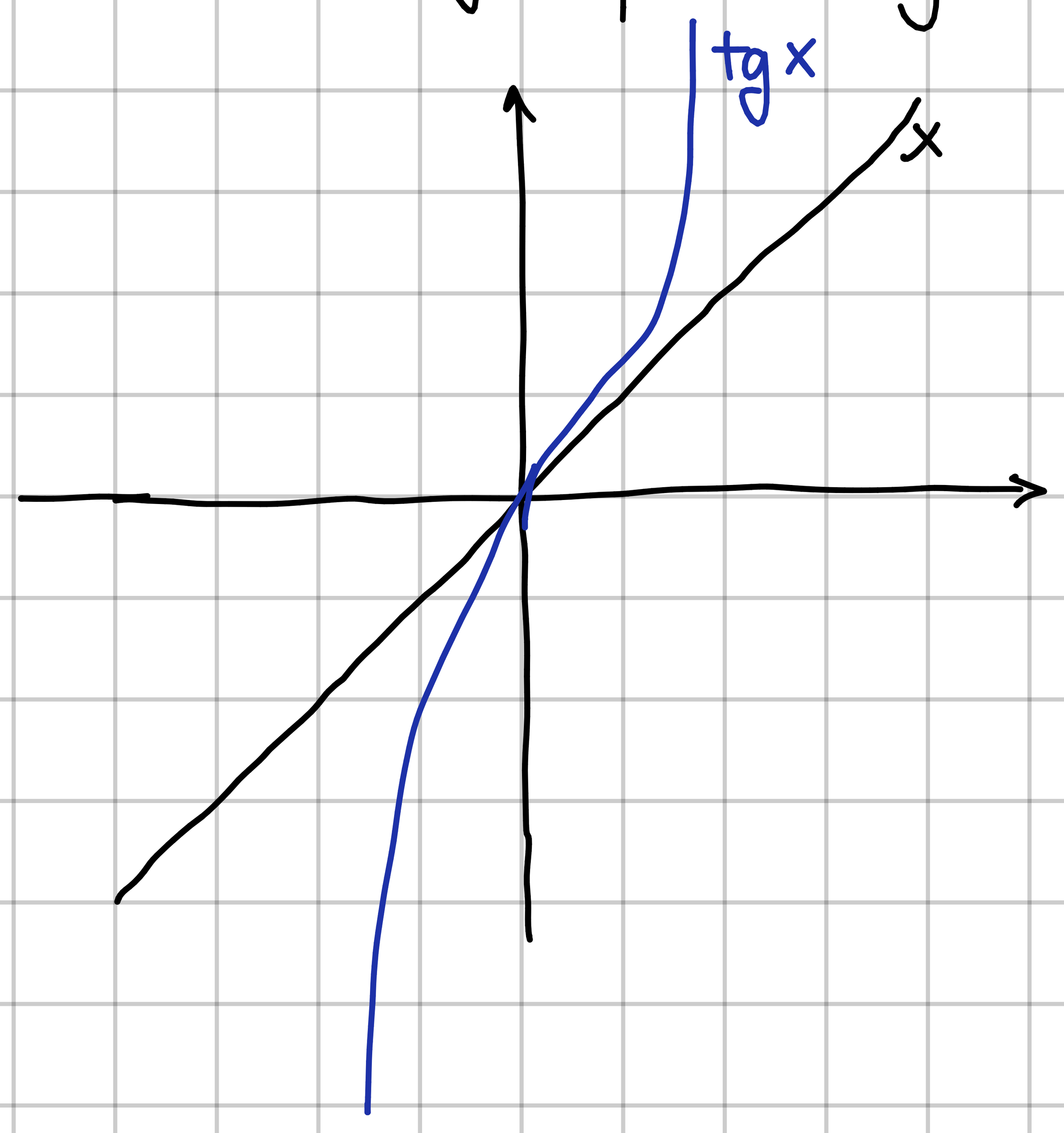
  $|g'(\alpha)|>1$: α odbojna

1. za rešitev enačbe $x-\operatorname{tg}x=0$ v okolici 0 imamo na voljo 2 iteracijski fun.

$$g_1(x)=\operatorname{tg}x$$
$$g_2(x)=\operatorname{arctg}x$$

Katera fun. je primernejša za iskanje negibne točke $\alpha=0$?

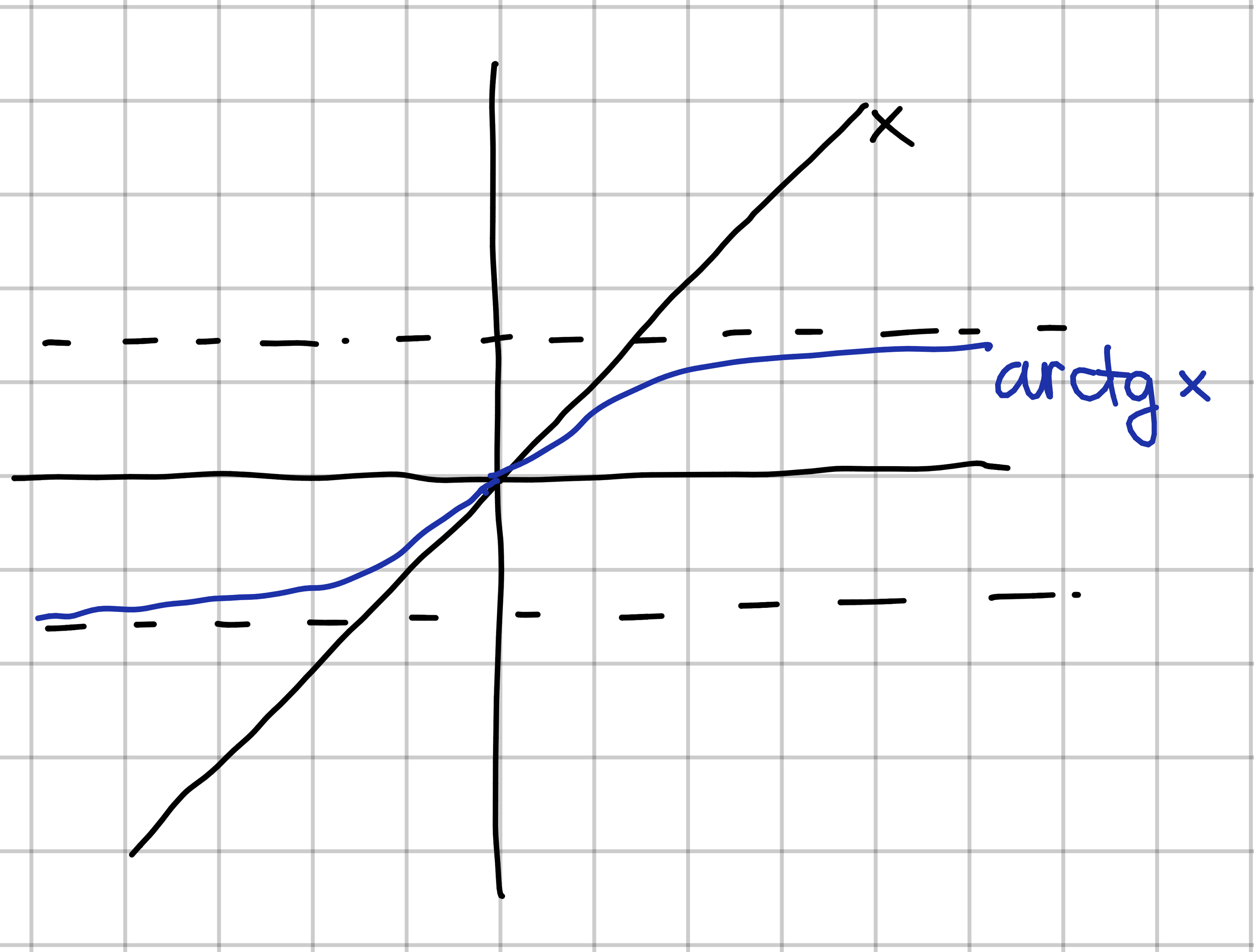
a)



$$\operatorname{tg}'x=\frac{1}{\cos^2 x} \qquad \operatorname{tg}'0=1$$

$$\operatorname{tg}'x=\begin{cases}1; & x=0\\ >1; & \text{sicer}\end{cases}$$

b)



$$\operatorname{arctg}'x=\frac{1}{1+x^2}$$

$$\operatorname{arctg}'0=1$$

$$\operatorname{arctg}'x=\begin{cases}1; & x=0\\ <1; & \text{sicer}\end{cases}$$

Naj bo α negibna točka in naj bo $I = [\alpha-\delta, \alpha+\delta]$ okolica točke α, za katero velja $|g'(\alpha)| < 1 \quad \forall x \in I$.
Potem bo zaporedje $(x_r)_r$, $x_{r+1} = g(x_r)$, konvergiralo k $\alpha \quad \forall x_0 \in I$.

(2.) Ničlo fun. $f(x) = x^5 - 10x + 1$ na intervalu $[0, 0.2]$ iščemo z iteracijsko funkcijo $g(x) = \dfrac{x^5+1}{10}$

a) Pokaži, da ima f na $[0, 0.2]$ natanko 1 ničlo.

$$f(0) = 0 - 0 + 1 = 1$$
$$f(0{,}2) = \left(\tfrac{1}{5}\right)^5 - 2 + 1 < 0$$
f je zvezna

$\Rightarrow$ f ima vsaj 1 ničlo na $[0, \frac{1}{5}]$

$f'(x) = 5x^4 - 10$ ali je to $< 0 \quad \forall x \in [0, \frac{1}{5}]$

$\rightsquigarrow$ ali je $x^4 < 2 \quad \forall x \in [0, \frac{1}{5}]$

$\downarrow$

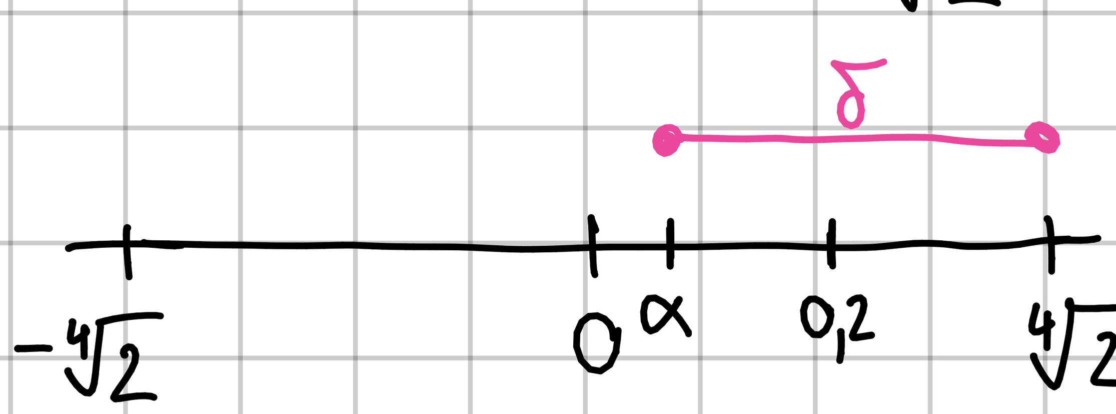
to očitno drži

$\Rightarrow$ f je padajoča

$\Rightarrow$ f ima natanko 1 ničlo na I

b) Utemeljite, da začetni približek $x_0 = 0$ zagotavlja konvergenco k tej ničli.

$$g'(x) = \frac{1}{10} \cdot (5x^4) = \frac{x^4}{2}$$
$$\left|\frac{x^4}{2}\right| < 1$$
$$\frac{x^4}{2} < 1 \quad \rightsquigarrow \quad x^4 < 2$$
$$-\sqrt[4]{2} < x < \sqrt[4]{2}$$



$$I = [\alpha - (\sqrt[4]{2} - \alpha), \alpha + (\sqrt[4]{2} - \alpha)]$$
$$= [2\alpha - \sqrt[4]{2}, \sqrt[4]{2}]$$

$x_0 \in I$ ✓

c) Pri $x_0 = 0$ izračunajte x_1 in x_2.

$$g(0) = \frac{1}{10} = x_1$$

$$g\left(\frac{1}{10}\right) = \frac{\left(\frac{1}{10}\right)^5 + 1}{10} = \frac{\frac{10^5+1}{10^5}}{10} = \frac{10^5+1}{10^6} = x_2$$

3. Iteracijska fun. $g(x) = -x^2 + 8x - 12$

a) Določite negibne točke in privlačnost / odbojnost.

$$-x^2 + 8x - 12 = x$$
$$x^2 - 7x + 12 = 0$$
$$(x-3)(x-4) = 0$$
$$x_1 = 3$$
$$x_2 = 4$$

negibni točki sta 3 in 4

$$g'(x) = -2x + 8$$

$$g'(3) = -6 + 8 = -2$$
$$|g'(3)| = 2 > 1$$
$\Rightarrow$ 3 je odbojna

$$g'(4) = -8 + 8 = 0$$
$$|g'(4)| = 0 < 1$$
$\Rightarrow$ 4 je privlačna

b) Za katere začetne približke imamo konvg. k privlačnim negibnim točkam zagotovljeno na podlagi odvoda fun. g?

$$|-2x+8| < 1 \quad \leftarrow |g'(x)| < 1$$

i) $-2x + 8 < 1$
$$-2x < -7$$
$$2x > 7$$
$$x > \frac{7}{2}$$

ii) $2x - 8 < 1$
$$2x < 9$$
$$x < \frac{9}{2}$$

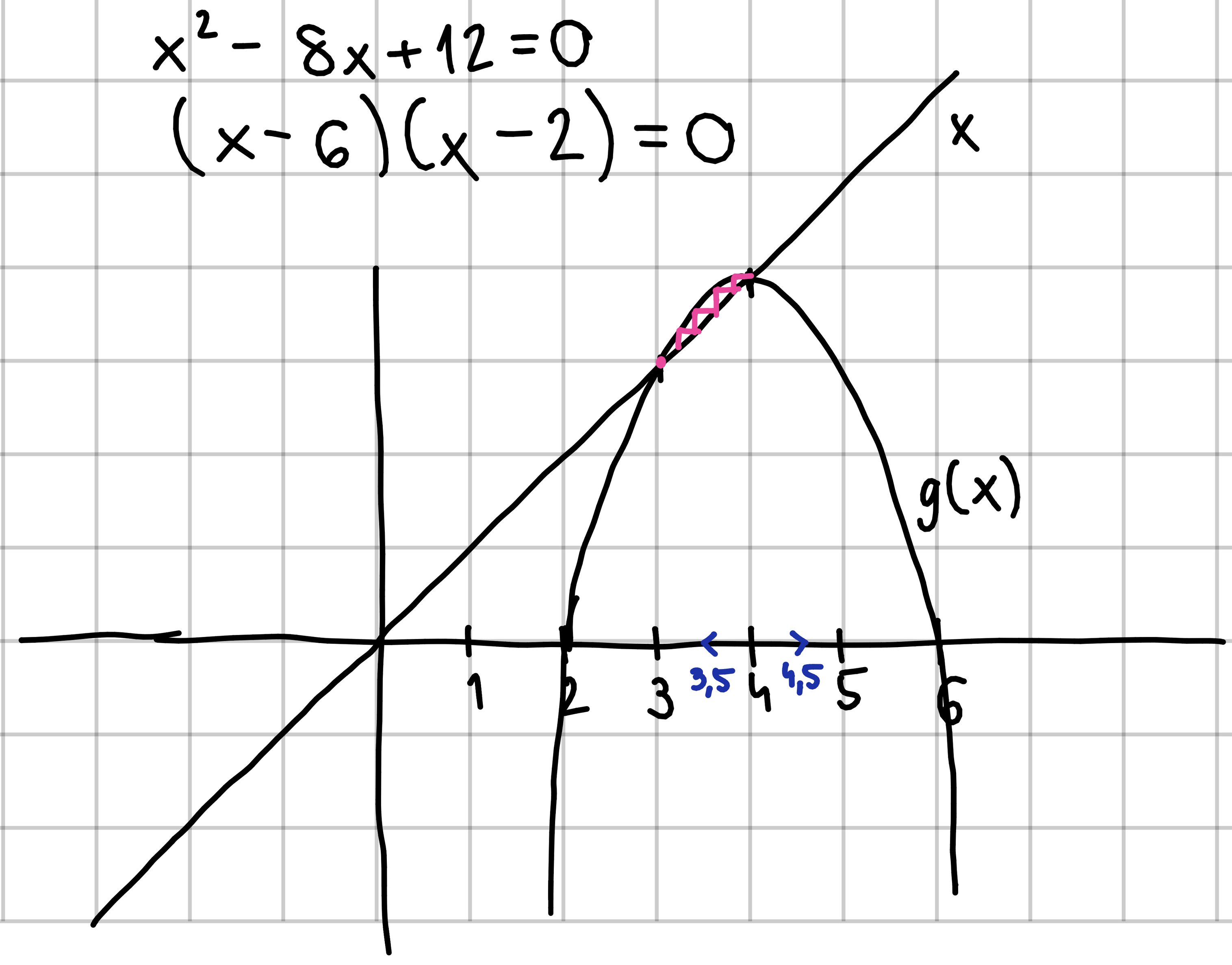
$$I = \left(\frac{7}{2}, \frac{9}{2}\right)$$ $\rightarrow$ to je simetričen interval okoli privlačne točke

↓ to mora biti

$\forall$ začetni približek iz I imamo konvg.

c) Za katere začetne približke iteracija dejansko konvergira?

$$-x^2 + 8x - 12 = 0$$
$$x^2 - 8x + 12 = 0$$
$$(x-6)(x-2) = 0$$



$$\forall x_0 \in (3,5) \Rightarrow \left(\lim_{r\to\infty} x_r = 4 \iff \lim_{r\to\infty} |x_r - 4| = 0 \right)$$

$$x_r - 4 = g(x_{r-1}) - 4 =$$
$$= -x_{r-1}^2 + 8x_{r-1} - 16 =$$
$$= -(x_{r-1}-4)^2 = ((x_{r-2}-4)^2)^2 = \dots$$

$$x_{r-1} - 4 = -(x_{r-2}-4)^2 \qquad = \underline{\underline{(x_0-4)^{2^r}}}$$

$$\lim_{r\to\infty} (x_0-4)^{2^r} = \begin{cases} 0 ; & |x_0-4| < 1 \\ \neq 0 ; & \text{sicer} \end{cases}$$

ker $x_0 \in (3,5) \Rightarrow$ limita je 0

6.11.2025

$$g(\alpha) = \alpha$$
$$x_{r+1} = g(x_r)$$

Red konvergence:

α negibna točka

$$g'(\alpha) = g''(\alpha) = \dots = g^{(p-1)}(\alpha) = 0$$
in $g^{(p)}(\alpha) \neq 0$

$\Rightarrow$ red je p

(1.) Pokažite, da lahko $\sqrt{a}$, $a \geq 0$ izračunamo z iteracijo

$$x_{r+1} = x_r \cdot \frac{x_r^2 + 3a}{3x_r^2 + a}$$

$$g(x) = x \cdot \frac{x^2+3a}{3x^2+a}$$
$$g(\sqrt{a}) = \sqrt{a} \cdot \frac{a+3a}{3a+a} = \sqrt{a}$$

- $\sqrt{a}$ negibna točka ✓
- $\sqrt{a}$ privlačna, $|g'(\sqrt{a})| < 1$

$$g'(x) = \frac{x^2+3a}{3x^2+a} + x \cdot \frac{2x(3x^2+a) - 6x(x^2+3a)}{(3x^2+a)^2}$$
$$= \frac{3x^4 + 3a^2 + 10x^2a + \cancel{6x^4} + 2x^2a - \cancel{6x^4} - 18x^2a}{(3x^2+a)^2} =$$

$$= \frac{3x^4 + 3a^2 - 6x^2a}{(3x^2+a)^2} = \frac{3(x^2-a)}{(3x^2+a)^2}$$

$$g'(\sqrt{a}) = \frac{3(a-a)}{(3a+a)^2} = 0$$

$\sqrt{a}$ je privlačna

b) Določite red konvergence

$$g''(x) = 3 \cdot \frac{2(x^2-a)\cdot 2x(3x^2+a)^2 - (x^2-a)^2 \cdot 2(3x^2+a)\cdot 6x}{(3x^2+a)^4} =$$

$$= 12x \frac{(x^2-a)(3x^2+a-3x^2+3a)}{(3x^2+a)^3} =$$

$$= \underbrace{\frac{48ax}{(3x^2+a)^3}}_{h_1(x)} \underbrace{(x^2-a)}_{h_2(x)}$$

vemo: $h_2(\alpha) = 0$

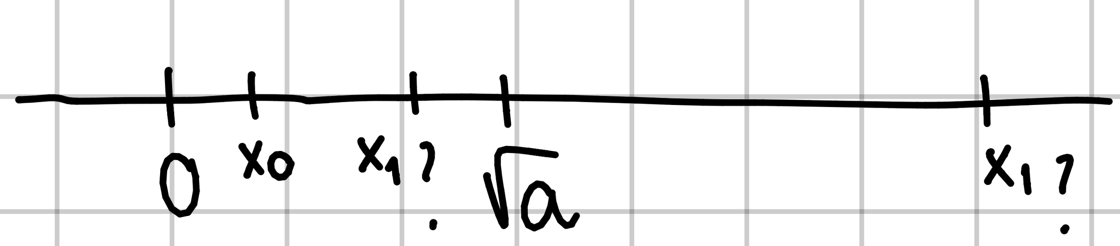
$$g''(\alpha) = \frac{48a\sqrt{a}(a-a)}{\sim} = 0$$

$$g'''(x) = \underbrace{h_1'(x)\cdot h_2(x)}_{\text{za } x=\sqrt{a} \text{ je to } =0} + \underbrace{h_1(x)\cdot h_2'(x)}$$

$$g'''(\alpha) = \frac{48a\sqrt{a}\cdot 2\sqrt{a}}{(3a+a)^3} \neq 0$$

$\Rightarrow$ red konvg. je kubičen (3)

c) Pokažite, da metoda konvergira $\forall x_0 > 0$



i) $x_0 \in (0, \sqrt{a})$

ii) $x_0 \in (\sqrt{a}, \infty)$

i) $$x_1 = x_0 \cdot \frac{x_0^2 + 3a}{3x_0^2 + a}$$

$x_1 < \sqrt{a}$?

$$x_0 \cdot \underbrace{\frac{x_0^2+3a}{3x_0^2+a}}_{\text{pozitivno}} \overset{?}{<} \sqrt{a}$$

$$x_0 \cdot (x_0^2 + 3a) \overset{?}{<} \sqrt{a}\,(3x_0^2 + a)$$

$$x_0^3 + 3ax_0 < 3x_0^2\sqrt{a} + a\sqrt{a}$$

$$x_0^3 - 3x_0^2\sqrt{a} + 3ax_0 - a\sqrt{a} < 0$$

$$(\underbrace{x_0 - \sqrt{a}}_{<0})^3 < 0$$

$$\Rightarrow (x_0 - \sqrt{a})^3 < 0$$

$\Rightarrow x_0 \in (0, \sqrt{a}) \Rightarrow x_1 < \sqrt{a}$

$x_1 > x_0$?

$$x_0 \cdot \frac{x_0^2 + 3a}{3x_0^2 + a} > x_0$$

$$x_0^3 + 3ax_0 > 3x_0^3 + x_0 a$$

$$2ax_0 > 2x_0^3$$

$$2ax_0 - 2x_0^3 > 0$$

$$ax_0 - \underbrace{x_0^3}_{>0} > 0$$

da ker $x_0 \in (0, \sqrt{a})$

$\Rightarrow x_1 \in (0, \sqrt{a})$

rekurzivno $\Rightarrow x_2 \in (x_1, \sqrt{a}) \ldots$

$\Rightarrow$ dobimo naraščajoče zap. x_r
in omejeno z $\sqrt{a}$
$\Rightarrow$ zap. je konvergentno

konvg. k $\sqrt{a}$, ker je negibna točka

ii) $x_0 \in (\sqrt{a}, \infty)$

$x_1 > \sqrt{a}$)
$x_1 < x_0$

$\Rightarrow$ padajoče zap. x_r
in omejeno z $\sqrt{a}$
$\Rightarrow$ zap. je konvergentno

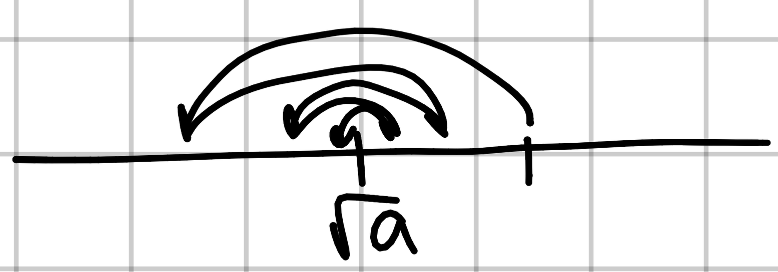
konvg. k $\sqrt{a}$, ker je negibna točka

preveriti še, da je $\sqrt{a}$ negibna točka na $(0, \infty)$

vse: $-\sqrt{a}, 0, \sqrt{a}$

$\Rightarrow \sqrt{a}$ edina na $(0, \infty)$

če bi skakali



pokazati: $|x_0 - \sqrt{a}| > |x_1 - \sqrt{a}|$

(2.) Enačbo $x^3 - A = 0$ rešujemo z nastavkom

$$x_{r+1} = \beta_1 x_1 + \beta_2 \cdot \frac{1}{x_r^2}, \quad r = 0, 1, 2, \dots$$

Določi β_1, β_2, da bo konvg. vsaj kvadratična. Ali je konvg. kubična?

→ konvg. ne sme biti linearna

$$g(x) = \beta_1 x + \beta_2 \cdot \frac{1}{x^2}$$

$$g'(x) = \beta_1 - 2\beta_2 \cdot \frac{1}{x^3}$$

$$x^3 - A = 0$$
$$x^3 = A$$
$$x = \sqrt[3]{A}$$
$$\alpha = \sqrt[3]{A}$$

$$g'(\sqrt[3]{A}) = \beta_1 - 2\beta_2 \cdot \frac{1}{(\sqrt[3]{A})^3} = \beta_1 - 2\beta_2 \cdot \frac{1}{A}$$

$$\beta_1 = 2\beta_2 \cdot \frac{1}{A}$$

$\sqrt[3]{A}$ mora biti še negibna

$$g(\sqrt[3]{A}) = \sqrt[3]{A}$$

$$\beta_1 \cdot \sqrt[3]{A} + \beta_2 \cdot \frac{1}{(\sqrt[3]{A})^2} = \sqrt[3]{A} \quad / \cdot (\sqrt[3]{A})^2$$

$$A \cdot \beta_1 + \beta_2 = A$$

$$A \cdot \beta_1 = A - \beta_2$$

$$\beta_1 = 1 - \frac{\beta_2}{A}$$

$$1-\frac{\beta_2}{A}=2\beta_2\cdot\frac{1}{A}$$

$$A-\beta_2=2\beta_2$$
$$A=3\beta_2$$
$$\beta_2=\frac{A}{3}$$

$$\beta_1=1-\frac{1}{3}=\frac{2}{3}$$

$$\Rightarrow g(x)=\frac{2}{3}x+\frac{A}{3x^2}$$

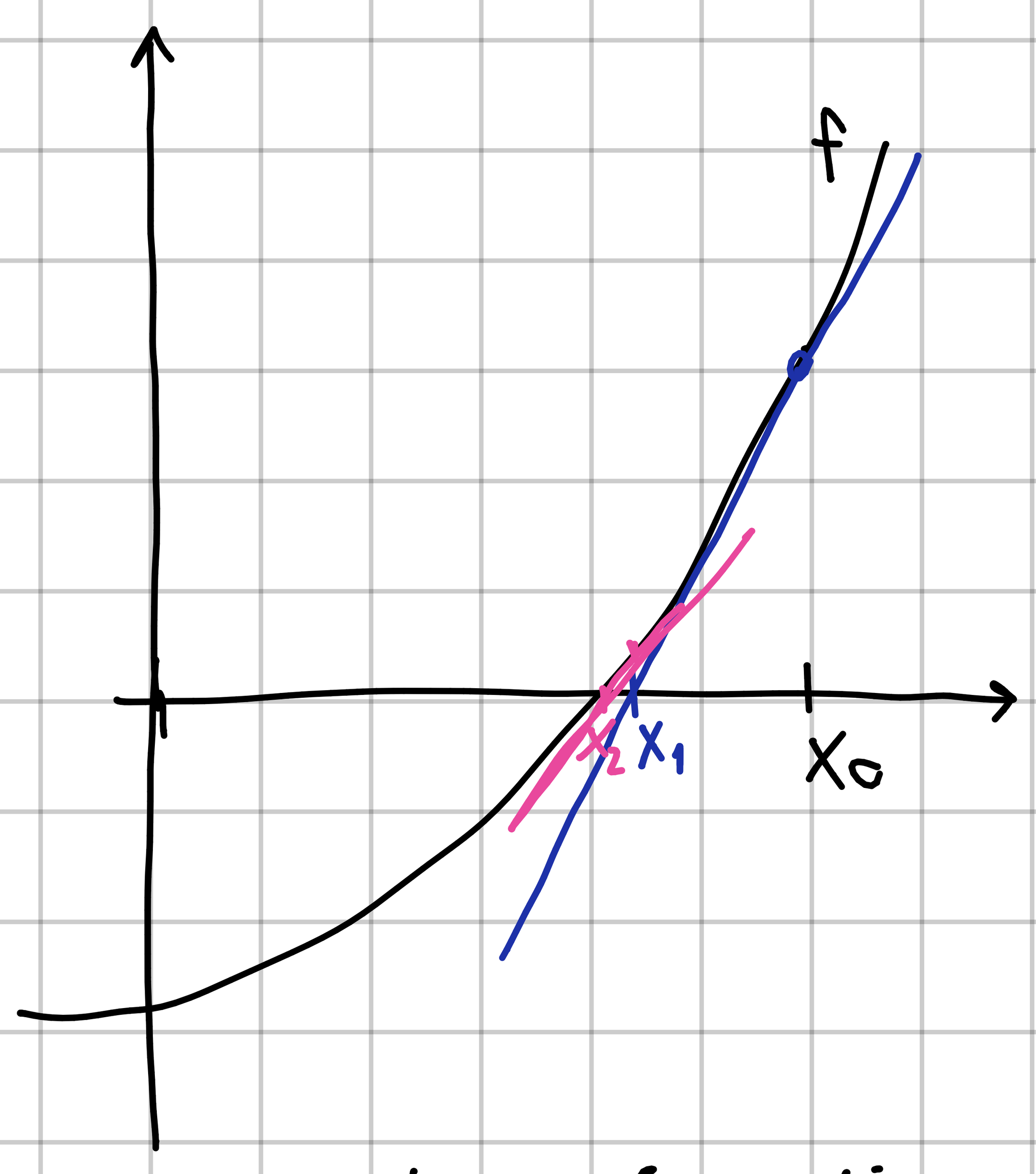
$$g'(x)=\frac{2}{3}-\frac{A}{3}\cdot\frac{2}{x^3}=\frac{2}{3}-\frac{2A}{3x^3}$$

$$g''(x)=\frac{3\cdot 2A}{3x^4}$$

$$g''(\sqrt[3]{A})=\frac{6A}{3(\sqrt[3]{A})^4}\neq 0$$

$\Rightarrow$ konvg. ni kubična

TANGENTNA METODA oz. Newtonova



navadna iteracija, kjer je it. fun.:

$$x_{r+1}=x_r-\frac{f(x_r)}{f'(x_r)}$$

③ a) Izpeljite metodo za računanje št. $\sqrt{a}$, $a>0$, s pomočjo tangentne metode.

poiščemo f, da bo njena ničla $\sqrt{a}$

$$f(x)=x^2-a$$
$$f'(x)=2x$$

$$x_{r+1} = x_r - \frac{x_r^2 - a}{2x_r}$$

b) Določi red konvergence

$$g(x) = x - \frac{x^2 - a}{2x}$$

$$g'(x) = 1 - \frac{2x \cdot 2x - 2(x^2 - a)}{4x^2} =$$

$$= 1 - \frac{4x^2 - 2x^2 + 2a}{4x^2} = 1 - \frac{x^2 + a}{2x^2} =$$

$$= \frac{x^2 - a}{2x^2}$$

$$g'(\sqrt{a}) = 0$$

$$g''(x) = \frac{2x \cdot 2x^2 - 4x(x^2 - a)}{4x^4} = \frac{4x^3 - 4x^3 + 4xa}{4x^4} =$$

$$= \frac{a}{x^3}$$

$$g''(\sqrt{a}) = \frac{a}{(\sqrt{a})^3} \neq 0$$

Red konvg. je 2.

c) Pokaži, da metoda konvg. $\forall x_0 > 0$

i) $x_0 \in (0, \sqrt{a})$

$$x_1 = x_0 - \frac{x_0^2 - a}{2x_0} \overset{?}{>} \sqrt{a}$$

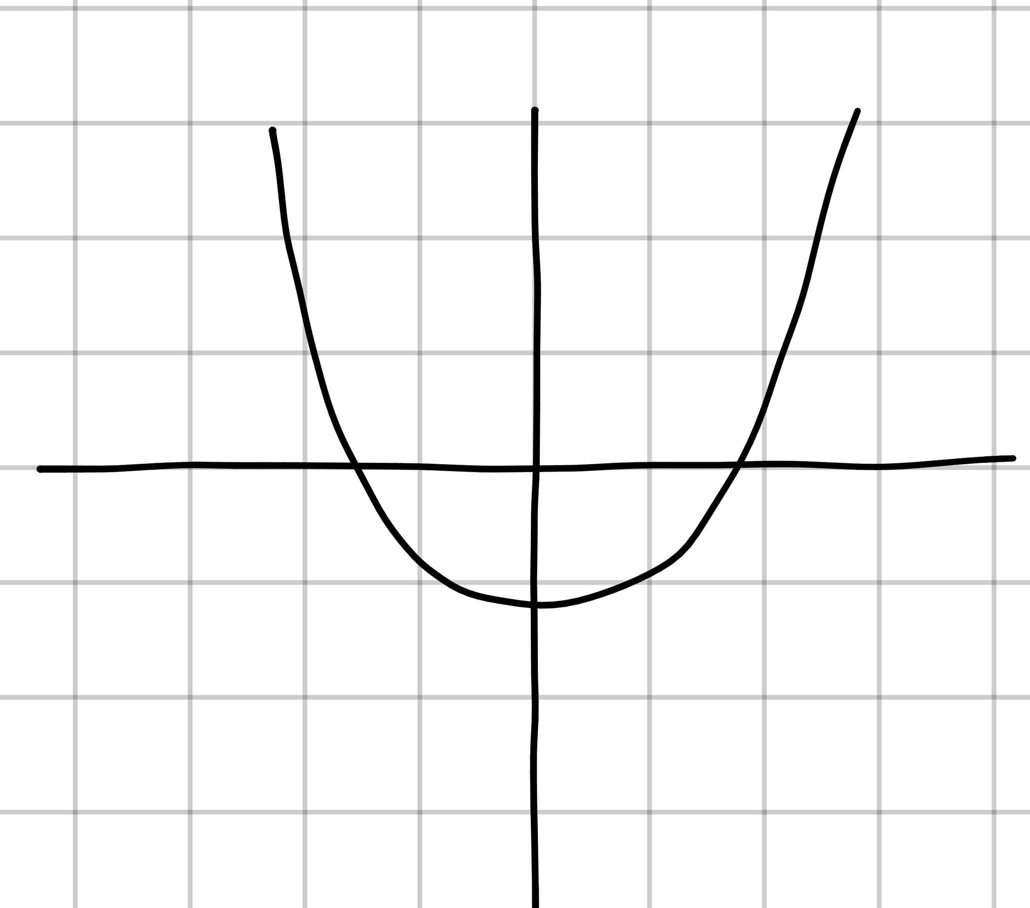
$$2x_0^2 - x_0^2 + a > 2x_0\sqrt{a}$$

$$x_0^2 - 2x_0\sqrt{a} + a > 0$$

$$(x_0 - \sqrt{a})^2 > 0$$

$\Downarrow$

$x_1 > \sqrt{a}$



ii) $x_0 \in (\sqrt{a}, \infty)$ $\Rightarrow$ $x_{r+1} < x_r$
$x_{r+1} > \sqrt{a}$

$x_1 > \sqrt{a}$

$x_1 < x_0$

$$x_0 - \frac{x_0^2 - a}{2x_0} < x_0$$

$$2x_0^2 - x_0^2 + a < 2x_0^2$$
$$a < x_0^2$$
$$\sqrt{a} < x_0$$

$$f(x) = x \cdot e^x + \frac{1}{2e}$$
$$f'(x) = e^x + x \cdot e^x$$
$$f''(x) = e^x + e^x + x \cdot e^x =$$
$$= 2e^x + x \cdot e^x$$

- $x_0 \in (0, \sqrt{a}) \Rightarrow x_1 > \sqrt{a}$
- $x_0 \in (\sqrt{a}, \infty)$

$$x_{r+1} < x_r$$
$$x_r > \sqrt{a}$$
$$(x_{r-1} - \sqrt{a})^2 > 0$$

13.11. 2025

1. Naj bo f 2x zv. odv. in α njena enostavna ničla.

a) Pokaži, da metoda $x_{r+1} = \underbrace{x_r - \frac{2f(x_r)f'(x_r)}{2f'(x_r)^2 - f(x_r)f''(x_r)}}_{g(x)}$ Halleyeva metoda

ustreza tangentni metodi za $F(x) = \frac{f(x)}{\sqrt{|f'(x)|}}$

$$g(x) = x - \frac{F(x)}{F'(x)}$$

$$F'(x) = \frac{f'(x) \cdot \sqrt{|f'(x)|} - f(x) \cdot \frac{1}{2\sqrt{|f'(x)|}} \cdot \frac{f'(x)}{|f'(x)|} \cdot f''(x)}{|f'(x)|}$$

$$F'(x) = \frac{f'(x)\left(\sqrt{|f'(x)|} - \frac{f(x)f''(x)}{2\sqrt{|f'(x)|^3}}\right)}{|f'(x)|}$$

$$F'(x) = \frac{f'(x)\left(\frac{2|f'(x)|^2 - f(x)f''(x)}{2\sqrt{|f'(x)|^3}}\right)}{|f'(x)|}$$

$$F'(x) = \frac{f'(x)(2f'(x)^2 - f(x)f''(x))}{2|f'(x)|^{5/2}}$$

$$g(x) = x - \frac{\frac{f(x)}{\sqrt{|f'(x)|}}}{\frac{f'(x) \cdot (2f'(x)^2 - f(x)f''(x))}{2|f'(x)|^{5/2}}}$$

$$g(x) = x - \frac{2f(x)f'(x)^2}{f'(x)(2f'(x^2) - f(x)f''(x))}$$

b) Poenostavi metodo za $f(x) = x^2 - a$

$$g(x) = x - \frac{2(x^2-a)\cdot 2x}{2(2x)^2 - (x^2-a)2} = x - \frac{2x^3 - 2ax}{4x^2 - x^2 + a} =$$

$$= \frac{x(3x^2+a) - (2x^3 - 2ax)}{3x^2+a} = \frac{x^3 + 3ax}{3x^2+a}$$

(2.) Obravnavaj red konvg. tangentne metode za fun. f, če je
- α njena enostavna ničla
- α njena m-kratna ničla (namig: Taylorjeva vrsta okoli α)

Če poznate kratnost ničle α, popravite tangentno metodo tako, da bo reda vsaj 2.

$\alpha \dots$ negibna točka

$$g'(\alpha) = \dots = g^{(p-1)}(\alpha) = 0$$
$$g^{(p)}(\alpha) \neq 0 \quad \Rightarrow \text{red konvg. je } p$$

$g(\alpha) = \alpha$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

$$g'(\alpha) = \frac{\overbrace{f(\alpha)}^{=0 \text{ zagotovo}} f''(\alpha)}{f'(\alpha)^2}$$

i) α enostavna ničla: $f(\alpha) = 0$
$f'(\alpha) \neq 0$

$\Rightarrow g'(\alpha) = 0$
$\Rightarrow$ red konvg. je vsaj 2

$$g''(x) = \frac{(\overbrace{f(x)}^{=0}f'''(x) + f'(x)f''(x))\cdot f'(x)^2 - 2\overbrace{f(x)}^{=0}(f''(x))^2 f'(x)}{(f'(x))^4}$$

$$\overset{x=\alpha}{=} \frac{f'(\alpha)^3 f''(\alpha)}{f'(\alpha)^4} = \frac{f''(\alpha)}{f'(\alpha)} \neq 0 \text{ v splošnem}$$

ii) α je m-kratna ničla $\rightarrow f(\alpha) = 0,\ f'(\alpha) = 0, \dots, f^{(m-1)}(\alpha) = 0$
$f^{(m)}(\alpha) \neq 0$

$$\lim_{x\to\alpha} \frac{f(x)f''(x)}{(f'(x))^2}$$

$$f(x) = \underbrace{f(\alpha)}_{=0} + \underbrace{f'(\alpha)}_{=0}(x-\alpha) + \frac{\overbrace{f''(\alpha)}^{=0}}{2}(x-\alpha)^2 + \dots + \underbrace{\overbrace{\frac{f^{(m)}(\alpha)}{m!}}^{\text{konst.}}(x-\alpha)^m}_{} + \underbrace{O((x-\alpha)^{m+1})}_{\text{ostanek}}$$

$$f'(x) = \frac{f^{(m)}(\alpha)}{m!} \cdot m(x-\alpha)^{m-1} + \sigma((x-\alpha)^m)$$

$$f''(x) = \frac{f^{(m)}(\alpha)}{(m-2)!} \cdot (x-\alpha)^{m-2} + \sigma((x-\alpha)^{m-1})$$

$$\frac{f(x)f''(x)}{f'(x)^2} = \frac{\left(\frac{f^{(m)}(\alpha)}{m!}(x-\alpha)^m + \sigma((x-\alpha)^{m+1})\right)\left(\frac{f^{(m)}(\alpha)}{(m-2)!}(x-\alpha)^{m-2} + \sigma((x-\alpha)^{m-1})\right)}{\left(\frac{f^{(m)}(\alpha)}{(m-1)!}(x-\alpha)^{m-1} + \sigma((x-\alpha)^m)\right)^2}$$

te grejo vsi proti 0

→ zanima nas člen z najnižjo potenco

$$= \frac{\frac{(f^{(m)}(\alpha))^2}{m!(m-2)!}(x-\alpha)^{2m-2} + \sigma((x-\alpha)^{2m-1})}{\frac{(f^{(m)}(\alpha))^2}{((m-1)!)^2}(x-\alpha)^{2m-2} + \sigma((x-\alpha)^{2m-1})}$$

$$\lim_{x\to\alpha} \frac{\frac{(f^{(m)}(\alpha))^2}{m!(m-2)!}(x-\alpha)^{2m-2} + \sigma((x-\alpha)^{2m-1})}{\frac{(f^{(m)}(\alpha))^2}{((m-1)!)^2}(x-\alpha)^{2m-2} + \sigma((x-\alpha)^{2m-1})} \quad \begin{matrix} /:(x-\alpha)^{2m-2} \\ /:(x-\alpha)^{2m-2} \end{matrix}$$

$$= \frac{((m-1)!)^2}{m!\,(m-2)!} = \frac{m-1}{m} = 1 - \frac{1}{m}$$

$$\Rightarrow \lim_{x\to\alpha} g'(\alpha) = 1 - \frac{1}{m}$$

$$m > 1 \Rightarrow \quad g'(\alpha) \neq 0$$

⇒ red konvg. je 1

$$g_1(x) = x - m \cdot \frac{f(x)}{f'(x)}$$

↳ popravljena metoda, preverimo, če je red vsaj 2

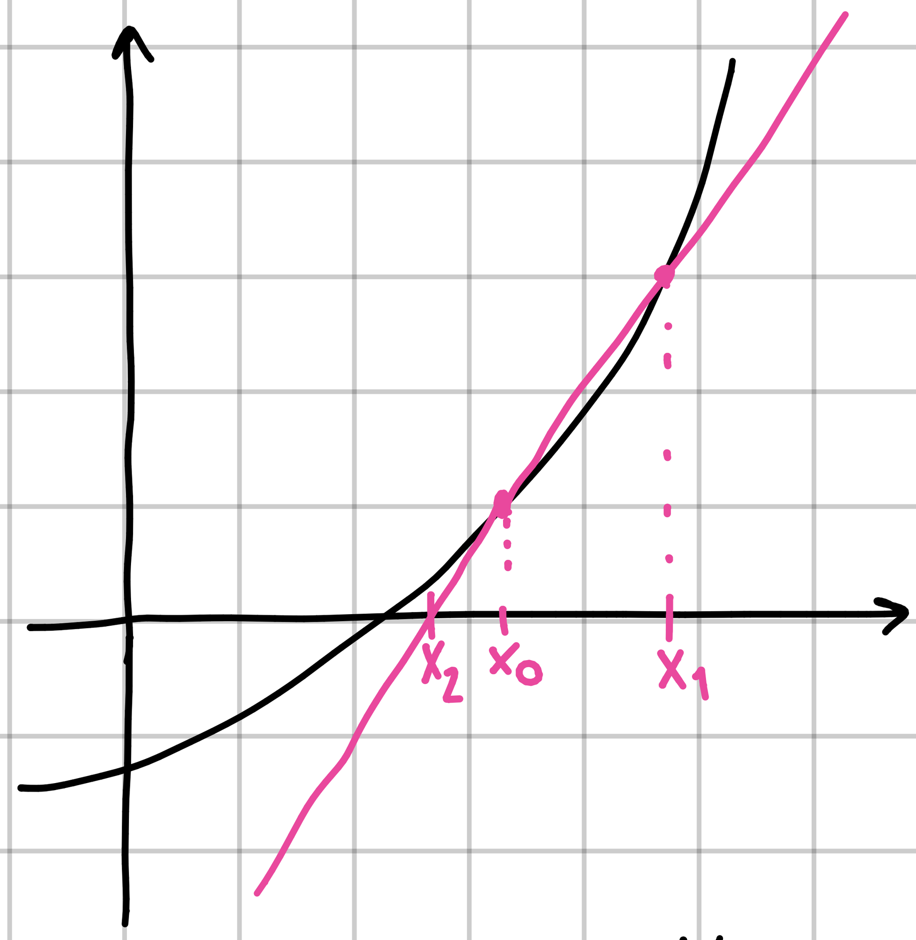
$$g_1'(x) = 1 - m \cdot \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} =$$

$$= \frac{(1-m)(f'(x))^2 + f(x)f''(x)}{f'(x)^2} = (1-m) + \frac{m\,f(x)f''(x)}{f'(x)^2}$$

$$\overset{x\to\alpha}{=} 1-m+m\left(1-\frac{1}{m}\right) = 1-m+m-1 = 0$$

⇒ red je vsaj 2

Sekantna metoda



$$x_{r+1} = x_r - \frac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})}$$

(3.) V nekem naselju danes ($x=0$) živi 2000 odraslih ljudi. V tem naselju je danes 1250 stanovanj. Za naslednjih 20 let bo št. ljudi opisovala funkcija $f_L(x) = 1000 + 1000 \cdot 2^{x/20}$. Oblasti načrtujejo gradnjo novih stanovanj s konstantno hitrostjo 10 stanovanj na leto. Zanima nas, kdaj bo št. oseb enako 2x št. stanovanj. Rešitev določite z enim korakom sekantne metode pri $x_0 = 20$ in $x_1 = 19$

$$f_L(x) = 2(1250 + 10x)$$

$$1000 + 1000 \cdot 2^{x/20} = 2500 + 20x$$

$$1000 \cdot 2^{x/20} = 1500 + 20x$$

$$f(x) = 1500 + 20x - 1000 \cdot 2^{x/20} = 0$$

$$f(20) = 1500 + 400 - 1000 \cdot 2 = $$
$$= 1900 - 2000 = -10$$
$$f(0) = 1500$$

$\Rightarrow$ imamo ničlo na intervalu $[0,20]$

$$\frac{f(x)}{20} = f(x) = 75 + x - 50 \cdot 2^{x/20}$$

$$x_2 = 19 - \frac{f(19)\cdot(19-20)}{f(19) - f(20)} =$$

$$= 19 + \frac{f(19)}{f(19)-f(20)} = 19 + \frac{f(19)}{f(19)+5} =$$

$$= 19 - \frac{2,6}{2,4} \approx 17,9$$

$f(19) \approx -$

$75 + 19 - 50 \cdot 2$

SISTEMI LINEARNIH ENAČB

4\. Dokaži, da ničle polinoma $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ ustrezajo lastnim vrednostim matrike

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix} \to n \times n \text{ matrika}$$

$$\det(\lambda I - A) = 0$$

indukcija:

$n=1$: $A_1 = [-a_0]$

$$\det(\lambda I - A_1) = \lambda + a_0 = 0$$
$$\lambda = -a_0$$

$$p_1(x) = x + a_0$$
$$p(-a_0) = 0$$

$n-1 \rightsquigarrow n$

$$\det(\underbrace{\lambda I - A_n}) = \lambda \det A_{11} + (-1)^{1+n} a_0 \det A_{1n}$$

$$\det A_{1n} = (-1)^{n-1}$$

$$\lambda I - A_n = \begin{bmatrix} \lambda & 0 & \dots & a_0 \\ 1 & \lambda & \dots & a_1 \\ 0 & 1 & \dots & a_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda + a_{n-1} \end{bmatrix}$$

A_{1n}

A_{11}

I.P.: $n-1$

$$\tilde{A} = \begin{bmatrix} 0 & 0 & & -a_1 \\ 1 & 0 & & \vdots \\ & & & \vdots \\ & & & -a_{n-1} \end{bmatrix}$$

$$p(x) = a_1 + x a_2 + \dots + a_{n-1}x^{n-2} + x^{n-1}$$

$$= a_0 + \lambda(a_1 + \lambda a_2 + \dots + a_{n-1}\lambda^{n-2} + \lambda^{n-1}) =$$

$$= a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_{n-1}\lambda^{n-1} + \lambda^n$$

Norme

- vektorske norme $x \in \mathbb{C}^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max_{i \in [n]} |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

- matrične norme : $A \in \mathbb{C}^{n \times n}$

$$\|A\|_1 = \max_{j \in [n]} \left(\sum_{i=1}^{n} a_{ij} \right) \quad \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\|A\|_\infty = \max_{i \in [n]} \left(\sum_{j=1}^{n} a_{ij} \right)$$

$$\|A^H\|_1 = \|A\|_\infty$$

$$\|A^H\|_\infty = \|A\|_1$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{tr(A^H A)}$$

$$\|A_2\| = \sigma_{max}(A) = \sqrt{\lambda_{max}(A^H A)}$$

5. a) Naj bo

$$A = \begin{array}{cc} \begin{array}{ccc} 9 & 6 & 6 \end{array} & \\ \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & -1 & 2 \end{bmatrix} & \begin{array}{c} 6 \\ 10 \\ 5 \end{array} \end{array}$$

Izračunaj $\|A\|_1$, $\|A\|_\infty$ in $\|A\|_F$.

$$\|A\|_1 = \max\{9, 6, 6\} = 9$$

$$\|A\|_\infty = \max\{6, 10, 5\} = 10$$

$$\|A\|_F = \sqrt{4+1+9+25+16+1+4+1+4} = \sqrt{65}$$

b)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Izračunaj $\|A\|_2$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix}$$

$$\det(\lambda I - A^T A) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 4-\lambda & -6 \\ 0 & -6 & 10-\lambda \end{vmatrix} =$$

$$= (\lambda-1)((\lambda-4)(\lambda-10) - 36) =$$
$$= (\lambda-1)(\lambda^2 - 14\lambda + 40 - 36) =$$
$$= (\lambda-1)(\lambda^2 - 14\lambda + 4)$$

$$\lambda_1 = 1$$

$$\lambda_{2,3} = \frac{14 \pm \sqrt{14^2 - 16}}{2} = \frac{14 \pm \sqrt{180}}{2} = 7 \pm 3\sqrt{5}$$

$$\lambda_1 = 1$$
$$\lambda_2 = 7 + 3\sqrt{5}$$
$$\lambda_3 = 7 - 3\sqrt{5}$$

$$\|A\|_2 = \max\{\lambda_1, \lambda_2, \lambda_3\} = \sqrt{7+3\sqrt{5}} \approx 3.7$$

6. VAJE – 20. 11. 2025

1. za $\|A\|_2$ dokaži spodnje neenakosti $(A \in \mathbb{C}^{n\times n})$

a) $\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F$

Namig: $\lambda_i \ldots$ lastna vrednost za A : $tr(A) = \sum_{i=1}^{n} \lambda_i$

$$\|A\|_2 = \sqrt{\max_\lambda (A^H A)}$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{tr(A^H A)}$$

$$\|A\|_2^2 \leq \|A\|_F^2 = tr(A^H A) = \sum_{i=1}^{n} \lambda_i$$

$\lambda_i \ldots$ e. vrednosti od matrike $A^H A$

$$\| \qquad \max_\lambda (A^H A) = \max_i \lambda_i$$

$\lambda_i \geq 0$, ker $A^H A$ poz. semidef.

$$\Rightarrow \sum_{i=1}^{n} \lambda_i - \lambda_{max} \geq 0$$

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2$$

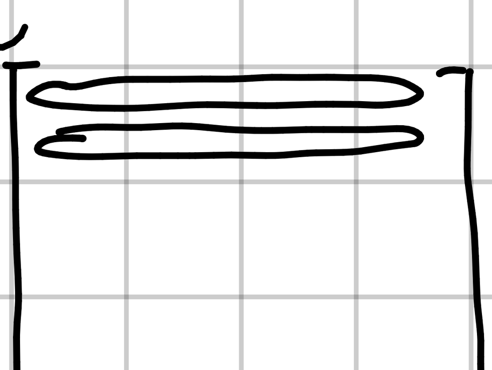
$$\frac{1}{n} \cdot \sum_{i=1}^{n} \lambda_i \overset{?}{\leq} \max_i \lambda_i$$

$$\sum_{i=1}^{n} \lambda_i \overset{?}{\leq} n \cdot \max_i \lambda_i \quad \checkmark$$

(vsako λ_i navzgor omejimo z $\max_i \lambda_i$)

b) $\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{n} \cdot \|A\|_\infty$

Namig: $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\, \|x\|_\infty$, $x \in \mathbb{C}^n$



$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \qquad \|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$\max_{x\neq 0} \frac{1}{\sqrt{n}} \frac{\overbrace{\|Ax\|_\infty}^{\leq \|Ax\|_2}}{\underbrace{\|x\|_\infty}_{\geq \frac{1}{\sqrt{n}}\|x\|_2}} \leq \max_{x\neq 0} \cancel{\frac{1}{\sqrt{n}}} \frac{\|Ax\|_2}{\cancel{\frac{1}{\sqrt{n}}} \cdot \|x\|_2} = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2$$

$$\|A\|_2 = \max_{x\neq 0} \frac{\overbrace{\|Ax\|_2}^{\leq \sqrt{n}\cdot\|Ax\|_\infty}}{\underbrace{\|x\|_2}_{\geq \|x\|_\infty}} \leq \max_{x\neq 0} \sqrt{n} \cdot \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\cdot\|A\|_\infty \quad \checkmark$$

c) $\frac{1}{\sqrt{n}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\, \|A\|_1$

Namig: $\|A^H\|_\infty = \|A\|_1$, $\|A^H\|_2 = \|A\|_2$

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

vemo: $\frac{1}{\sqrt{n}} \|A^H\|_\infty \leq \|A^H\|_2 \leq \sqrt{n}\cdot \|A^H\|_\infty$

$$\frac{1}{\sqrt{n}} \cdot \|A\|_1 \leq \|A\|_2 \leq \sqrt{n} \cdot \|A\|_\infty$$

d) $\|A\|_2 \leq \sqrt{\|A\|_1 \cdot \|A\|_\infty}$

Namig: Lema: $\forall$ matrično normo in poljubno lastno vrednost λ matrike A velja: $|\lambda| \leq \|A\|$

$$\|A\|_2^2 = \underbrace{\max_\lambda (A^H A)}_{\lambda_{max}(A^H A)} \leq \|A^H A\|_1 \leq \|A^H\|_1 \cdot \|A\|_1 = \|A\|_\infty \cdot \|A\|_1$$

submultiplikativnost

lema: $\lambda_{max}(A^H A) \leq \|A^H A\|_1$

2. Naj bosta $x, y \in \mathbb{R}^n$, $x, y \neq 0$. Naj bo $A = xy^T$. Izračunaj $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$ in $\|A\|_2$.

$\|xy^T\|_1 = ?$

$x = (x_1, \ldots, x_n)^T$
$y = (y_1, \ldots, y_n)^T$

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$

$\|x\|_\infty = \max_i |x_i|$

$$A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & & & \vdots \\ \vdots & & & \vdots \\ x_n y_1 & \cdots & \cdots & x_n y_n \end{bmatrix}$$

$$\|xy^T\|_1 = \max_j \sum_{i=1}^n |x_i y_j| = \max_j \sum_{i=1}^n |x_i| \cdot |y_j| = \max_j |y_j| \cdot \|x\|_1$$
$$= \|y\|_\infty \cdot \|x\|_1$$

$$\|A\|_1 = \|x\|_1 \cdot \|y\|_\infty$$

$$\|xy^T\|_\infty = \max_i \sum_{j=1}^n |x_i y_j| = \max_i \sum_{j=1}^n |x_i| \cdot |y_j| = \|y\|_1 \cdot \|x\|_\infty$$

$$\|A\|_\infty = \|x\|_\infty \cdot \|y\|_1$$

$$\|A\|_F^2 = \|xy^T\|_F^2 = \sum_{i=1}^n \sum_{j=1}^n |x_i y_j|^2 = \sum_{i=1}^n \sum_{j=1}^n x_i^2 y_j^2 = \sum_{i=1}^n x_i^2 \cdot \sum_{j=1}^n y_j =$$
$$= \|x\|_2^2 \cdot \|y\|_2^2$$

$$\|A\|_F = \|x\|_2 \cdot \|y\|_2$$

$$\|xy^T\|_2^2 = \max_\lambda (A^T A) = \|x\|_2^2 \cdot \|y\|_2^2$$

$$A^T A = (xy^T)^T xy^T =$$
$$= y \underbrace{x^T x}_{\|x\|_2^2} y^T = y \|x\|_2^2 y^T =$$
$$= \|x\|_2^2 \cdot \begin{bmatrix} y_1 y_1 & \dots & y_1 y_n \\ \vdots & & \\ y_n y_1 & \dots & y_n y_n \end{bmatrix} = \|x\|_2^2 \cdot \begin{bmatrix} y_1 \cdot y^T \\ \vdots \\ y_n \cdot y^T \end{bmatrix}$$

$$\Downarrow$$

rang$(yy^T) = 1$
$\Rightarrow$ samo 1 neničelna
l. vrednost

! $\boxed{\begin{array}{l} x^T x = \|x\|_2^2 \\ x x^T = \text{matrika} \end{array}}$

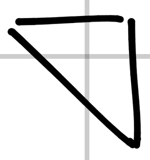
$\|y\|_2^2$ l. vrednost za yy^T

$$\downarrow$$

edina neničelna
l. vrednost
$\Rightarrow \|y\|_2^2 = \max_\lambda (yy^T)$

$Ax = \lambda x$
$yy^T \_\_ = \lambda \_\_$
$\lambda = \|y\|_2^2$
$(yy^T) y = \|y\|_2^2 \cdot y$

LU razcep

$A = LU$, $L =$ , $U =$ , $\frac{2}{3} n^3 + O(n^2)$

$Ax = b$
$L \underbrace{Ux}_{y} = b$

1.) $Ly = b$ prema substitucija (n^2)
2.) $Ux = y$ obratna substitucija $(n^2 + n)$

3. a) Določite LU razcep matrike

$$A = \begin{bmatrix} 2 & 1 & 3 & -4 \\ -4 & -1 & -4 & 7 \\ 2 & 3 & 5 & -3 \\ -2 & -2 & -7 & 9 \end{bmatrix}$$

• prvo vrstico prepišemo

$$-1-1(-2)$$

$$\begin{bmatrix} 2 & 1 & 3 & -4 \\ -2 & 1 & 2 & -1 \\ 1 & 2 & 2 & 1 \\ -1 & -1 & -4 & 5 \end{bmatrix}$$

• $-4-(-2)3=2$

$7-(-4)(-2)$

• ostalo v 1. stolpcu delimo z diagonalnim
• element v „novem A" = element v „starem A" − (el. v novem A skrajno zgoraj)·(el. v novem A skrajno levo)

$$\begin{bmatrix} 2 & 1 & 3 & -4 \\ -2 & 1 & 2 & -1 \\ 1 & 2 & -2 & 3 \\ -1 & -1 & -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 & -4 \\ -2 & 1 & 2 & -1 \\ 1 & 2 & -2 & 3 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 3 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) Preko LU razcepa rešite sistem enačb:

$$2x + 2y - 4z + w = -2$$
$$2z + x = 9$$
$$2y + x - w = 1$$
$$y - z + 1 = 0$$

$$\begin{bmatrix} 2 & 2 & -4 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & -4 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & -4 & 1 \\ 1/2 & -1 & 4 & -1/2 \\ 1/2 & 1 & 2 & -3/2 \\ 0 & 1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & -4 & 1 \\ 1/2 & -1 & 4 & -1/2 \\ 1/2 & -1 & 6 & -2 \\ 0 & -1 & 3 & -1/2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 2 & -4 & 1 \\ 1/2 & -1 & 4 & -1/2 \\ 1/2 & -1 & 6 & -2 \\ 0 & -1 & 1/2 & 1/2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & -1 & 1 & 0 \\ 0 & -1 & 1/2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 2 & -4 & 1 \\ 0 & -1 & 4 & -1/2 \\ 0 & 0 & 6 & -2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & -1 & 1 & 0 \\ 0 & -1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 1 \\ -1 \end{bmatrix}$$

$$y_1 = -2$$

$$\frac{1}{2} y_1 + y_2 = 9$$
$$-1 + y_2 = 9$$
$$y_2 = 10$$

$$\frac{1}{2} \cdot (-2) + (-1) \cdot 10 + 1 \cdot y_3 = 1$$
$$-1 - 10 + y_3 = 1$$
$$y_3 = 12$$

$$(-1) \cdot 10 + \frac{1}{2} \cdot 12 + y_4 = -1$$
$$-10 + 6 + y_4 = -1$$
$$y_4 = 3$$

$Ux = y$

$$\begin{bmatrix} 2 & 2 & -4 & 1 \\ 0 & -1 & 4 & -1/2 \\ 0 & 0 & 6 & -2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 12 \\ 3 \end{bmatrix}$$

$$\frac{1}{2} w = 3$$
$$w = 6$$

$$6z - 2 \cdot 6 = 12$$
$$6z = 24$$
$$z = 4$$

$$-1y + 4 \cdot 4 - \frac{1}{2} \cdot 6 = 10$$
$$-y + 16 - 3 = 10$$
$$y = 3$$

$$2x + 2 \cdot 3 - 4 \cdot 4 + 6 = -2$$
$$2x - 10 + 6 = -2$$
$$2x = 2$$
$$x = 1$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix}$$

Delno pivotiranje: dovoljujemo zamenjave vrstic

$$PA = LU$$

Kompletno pivotiranje: dovoljujemo zamenjave vrstic in stolpcev

$$PAQ = LU$$

4. Naj bo

$$A = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 2 & 1 & -4 & 2 \\ 3 & -2 & 3 & -1 \\ -1 & 3 & -1 & 1 \end{bmatrix}$$

Izračunaj LU razcep z delnim pivotiranjem in $\det(A)$.

$$\begin{bmatrix} 3 & -2 & 3 & -1 \\ 2 & 1 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 3 & -1 \\ 2/3 & 7/3 & -6 & 8/3 \\ 2/3 & 7/3 & -4 & 5/3 \\ -1/3 & 7/3 & 0 & 2/3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} -3 & -2 & 3 & -1 \\ 2/3 & 7/3 & -6 & 8/3 \\ 2/3 & 1 & 2 & -1 \\ -1/3 & 1 & 6 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -2 & 3 & -1 \\ 2/3 & 7/3 & -6 & 8/3 \\ -1/3 & 1 & 6 & -2 \\ 2/3 & 1 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -2 & 3 & -1 \\ 2/3 & 7/3 & -6 & 8/3 \\ -1/3 & 1 & 6 & -2 \\ 2/3 & 1 & 1/3 & -1/3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ -1/3 & 1 & 1 & 0 \\ 2/3 & 1 & 1/3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & -2 & 3 & -1 \\ 0 & 7/3 & -6 & 8/3 \\ 0 & 0 & 6 & -2 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\det A = \frac{\det L \cdot \det U}{\det P} = \frac{1 \cdot (-14)}{1} = -14$$

$\det L = 1$

$\det U = 3 \cdot \frac{7}{3} \cdot 6 \cdot (-\frac{1}{3}) = -14$

$\det P = (-1)^2$

$2 = \#$ zamenjav vrstic

(5.) Naj bo

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix}$$

Izračunajte LU razcep s kompletnim pivotiranjem in rešite sistem

$$Ax = \begin{bmatrix} 8 \\ 14 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & -6 \\ -3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} -3 & 2 & 1 \\ -6 & 2 & 4 \\ 3 & 3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -6 & 2 & 4 \\ -3 & 2 & 1 \\ 3 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} -6 & 2 & 4 \\ 1/2 & 1 & -1 \\ -1/2 & 4 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} -6 & 2 & 4 \\ -1/2 & 4 & -1 \\ 1/2 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} -6 & 2 & 4 \\ -1/2 & 4 & -1 \\ 1/2 & 1/4 & -3/4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1/4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} -6 & 2 & 4 \\ 0 & 4 & -1 \\ 0 & 0 & -3/4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A = P^{-1} L U Q^{-1}$$

$$Ax = b$$

$$(P^{-1} L U Q^{-1}) x = b$$

$$\underbrace{L U \underbrace{Q^{-1} x}_{z}}_{y} = Pb$$

1.) $Ly = Pb$
2.) $Uz = y$
3.) $Q^{-1} x = z$
$x = Qz$

1.) $$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 8 \end{bmatrix}$$

$y_1 = 14$
$y_2 = 7$
$y_3 = 8 - \frac{7}{4} - 7 = -\frac{3}{4}$

2.) $$\begin{bmatrix} -6 & 2 & 4 \\ 0 & 4 & -1 \\ 0 & 0 & -3/4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \\ -3/4 \end{bmatrix}$$

$$z_3 = 1$$
$$4z_2 = 7 + 1$$
$$z_2 = 2$$
$$z_1 = -1$$

3.) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = X$$

7. VAJE - 27. 11. 2025

1. Določite LU razcep za

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & 5 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

b, c, a

Kaj opazite?

$$\to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4/1 & -7 & 2 & 0 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & -2/7 & 11/7 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 4 & -7 & 2 & 0 \\ 0 & -2/7 & 11/7 & 6 \\ 0 & 0 & 7/11 & -20/11 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & -2/7 & 1 & 0 \\ 0 & 0 & 7/11 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 11/7 & 6 \\ 0 & 0 & 0 & -20/11 \end{bmatrix}$$

l, b, u

$$l_i = \frac{c_i}{u_i} \qquad i = 1, \dots, n-1$$

$$4 = \frac{4}{1}$$

$$-\frac{2}{7} = \frac{2}{-7}$$

$$\frac{7}{11} = \frac{1}{\frac{11}{7}} =$$

b) s pomočjo a) zapišite algoritem za LU razcep splošne tridiagonalne matrike:

$$A = \begin{bmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & b_3 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}$$

$$\begin{matrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & a_{32} & \ddots & \ddots & \\ & & \ddots & a_{n-1\,n-1} & a_{n-1\,n} \\ & & & a_{n\,n-1} & a_{nn} \end{matrix}$$

i) Preštejte št. operacij

$$\begin{bmatrix} a_1 & b_1 & & & 0 \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & b_3 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix} \rightarrow \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 \\ c_1/a_1 & a_2 - \frac{c_1}{a_1} b_2 & \dots & & 0 \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

2.) $l_1 = \frac{c_1}{a_1} \rightsquigarrow l_i = \frac{c_i}{u_i} \quad i = 1, \dots, n-1$

1.) $u_1 = a_1$

3.) $u_2 = a_2 - b_1 \cdot l_1 \rightsquigarrow u_i = a_i - b_{i-1} \cdot l_{i-1}$
$i = 2, \dots, n$

$u_1 = a_1$
for $i = 1 : n-1$
  $c_i = c_i / u_i$
  $u_{i+1} = a_{i+1} - b_i \cdot l_i$
end for

$\rightsquigarrow$ Thomasov algoritem

operacije: $3(n-1) = 3n-3$

ii) zapišite postopek za reševanje sistema $Ax = z$ in preštejte št. operacij.

$\underbrace{LUx}_{y} = z$

$Ux = y$

$Ly = z$

$Ly = z$: $\begin{bmatrix} 1 & & & \\ l_1 & 1 & & \\ & \ddots & \ddots & \\ & & l_{n-1} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$

$$1 \cdot y_1 = z_1$$
$$l_1 y_1 + 1 \cdot y_2 = z_2 \rightarrow y_2 = z_2 - l_1 y_1$$
$$l_2 y_2 + 1 \cdot y_3 = z_3$$
$$\vdots$$
$$l_i y_i + 1 \cdot y_{i+1} = z_{i+1}$$
$$\Big\} \; n-1$$

$$(n-1) \cdot 2 = 2n - 2$$

$$Ux = y : \begin{bmatrix} u_1 & b_1 & & \\ & u_2 & b_2 & \\ & & \ddots & b_{n-1} \\ & & & u_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

$$u_n x_n = y_n \rightarrow x_n = \frac{y_n}{u_n}$$

$$u_i x_i + b_i x_{i+1} = y_i \rightarrow x_i = \frac{y_i - b_i x_{i+1}}{u_i} \Big\} (n-1)3$$

$$3n - 3 + 1 = 3n - 2$$

(2.) Dana je nesingularna matrika $A \in \mathbb{R}^{n \times n}$ skupaj z inverzom A^{-1}. Zapišite postopek za izračun inverza

$$B = \begin{bmatrix} A & u \\ v^T & \alpha \end{bmatrix} \qquad u, v \in \mathbb{R}^n \setminus \{0\}, \; \alpha \in \mathbb{R}$$

Preštejte št. operacij.
Kdaj inverz sploh obstaja?

$$B^{-1} = \begin{bmatrix} C & y \\ x^T & \beta \end{bmatrix}$$

$$B \cdot B^{-1} = I$$

$$\begin{bmatrix} A & u \\ v^T & \alpha \end{bmatrix} \cdot \begin{bmatrix} C & y \\ x^T & \beta \end{bmatrix} = \begin{bmatrix} [AC + u x^T]_{n \times n} & [Ay + \beta u]_{n \times 1} \\ [v^T C + \alpha x^T]_{1 \times n} & [v^T y + \alpha \beta]_{1 \times 1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

$$v^T y + \alpha \beta = 1$$
$$Ay + \beta u = 0 \qquad y = -A^{-1} \beta u$$
$$v^T C + \alpha x^T = 0$$
$$AC + u x^T = I$$

$$v^T(-A^{-1} \beta u) + \alpha \beta = 1$$
$$\beta(-v^T A^{-1} u) + \beta \alpha = 1$$
$$\beta(\underbrace{-v^T}_{[\;]} \underbrace{A^{-1} u}_{[\;]} + \alpha) = 1 \qquad = [\;]$$

$$\beta = \frac{1}{-v^T A^{-1} u + \alpha}$$

1.) izračunamo $A^{-1}u = w$

2.) izračunamo $\beta = \frac{1}{-v^T w + \alpha}$

3.) $y = -\beta w$

$$AC = I - u x^T$$
$$C = A^{-1} - A^{-1} u x^T$$

$$v^T C + \alpha x^T = 0$$
$$v^T (A^{-1} - A^{-1} u x^T) + \alpha x^T = 0$$
$$v^T A^{-1} - v^T A^{-1} u x^T + \alpha x^T = 0$$
$$(-v^T A^{-1} u + \alpha) x^T = -v^T A^{-1}$$
$$x^T = \frac{-v^T A^{-1}}{-v^T A^{-1} u + \alpha} = -\beta (v^T A^{-1})$$

4) $x^T = -\beta (v^T A^{-1})$

5) $C = A^{-1} - w x^T$

$j = 1 + i$

$$\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} -2 & 4 & 0 & 0 & 0 \\ 4 & -4 & 3 & 0 & 0 \\ & & -6 & 2 & 0 \\ & & & -8 & 1 \\ & & & & -10 \end{bmatrix}
\end{array}$$

$n = 5$