Druga domača naloga Analize 3 za IŠRM

5. januar 2025

Povzetek

Za boljšo preglednost na računalnik prepisane rešitve. Zgolj kot dokaz reševanja so priložene še slike rokopisa.

Naloga 1. [1%] Naj bo K odsek krivulje, podane z

$$x^2 + y^2 = 1 \quad \text{in} \quad x \tan z = y$$

od točke $(1, 0, 0)$ do točke $(1, 0, 2\pi)$. Izračunaj integral

$$\int_K y\,dx + x\,dy + z\,dz.$$

Rešitev. $x^2 + y^2 = 1 \wedge x \tan z = y \Longrightarrow$

$$y = \sqrt{1 - x^2} \wedge z = \arctan \frac{y}{x} \Longrightarrow z = \left| \pm \arctan \frac{\sqrt{1 - x^2}}{x} \right|$$

Vpeljimo cilindrično parametrizacijo krivulje K:

$$x = 1 \cdot \cos\varphi, y = 1 \cdot \sin\varphi, z = \arctan \frac{\sqrt{1 - \cos^2\varphi}}{\cos\varphi} = \arctan \frac{\sin\varphi}{\cos\varphi} = \varphi$$

$\vec{r}\,(t) = (\cos\varphi, \sin\varphi, \varphi)$ za $\varphi \in [0, 2\pi]$ (omejitev iz navodila).

$$\dot{\vec{r}} = (-\sin\varphi, \cos\varphi, 1)$$

$$\int_K y\,dx + x\,dy + z\,dz = \int_K \vec{V}\left(\vec{S}\right)\, d\vec{S} = \cdots,$$

kjer je $\vec{V} : \mathbb{R}^3 \to \mathbb{R}^3$ s predpisom $(x, y, z) \mapsto (y, x, z)$.

$$\cdots = \int_0^{2\pi} \vec{V}\left(\vec{r}\,(\varphi)\right) \cdot \dot{\vec{r}}\,(\varphi)\ d\varphi = \int_0^{2\pi} (\sin\varphi, \cos\varphi, \varphi) \cdot (-\sin\varphi, \cos\varphi, 1)\ d\varphi =$$

$$= \int_0^{2\pi} \cos^2\varphi - \sin^2\varphi + \varphi\,d\varphi = \cancel{\int_0^{2\pi} \cos(2\varphi)\ d\varphi} + \left.\frac{\varphi^2}{2}\right|_0^{2\pi} =$$

$$= 2\pi^2$$

Naloga 2. [1%] Dokaži, da sta za krivuljo $K \subseteq \mathbb{R}^3$ naslednji trditvi ekvivalentni:

- krivulja K leži na neki sferi s središčem v izhodišču;
- tangenta v vsaki točki na krivulji K je pravokotna na vektor med izhodiščem in to točko.

Rešitev. Dokazujemo ekvivalenco. Naj bo $\vec{r} : [a, b] \to \mathbb{R}^3$ za $a, b \in \mathbb{R}$ parametrizacija krivulje K in $\vec{r}(t) =: (x(t), y(t), z(t))$.

($\Rightarrow$) Dokazujemo $\forall t \in [a, b] : \vec{r}(t) \cdot \dot{\vec{r}}(t) \stackrel{?}{=} 0$. Naj bo t poljuben. Po predpostavki K leži na neki sferi s središčem v izhodišču. Brez škode za splošnost naj je radij te sfere 1, torej $z(t) = \sqrt{1 - x^2(t) - y^2(t)}$.

$$\dot{\vec{r}}(t) = \left(\dot{x}(t), \dot{y}(t), \left(\sqrt{1 - x^2(t) - y^2(t)}\right)' = \frac{\left(1 - x^2(t) - y^2(t)\right)'}{2\sqrt{1 - x^2(t) - y^2(t)}}\right)$$

$$\vec{r}(t) \cdot \dot{\vec{r}}(t) =$$

$$= \left(x(t), y(t), \sqrt{1 - x^2(t) - y^2(t)}\right) \cdot \left(\dot{x}(t), \dot{y}(t), \frac{-\cancel{2}x(t)\dot{x}(t) - \cancel{2}y(t)\dot{y}(t)}{\cancel{2}\sqrt{1 - x^2(t) - y^2(t)}}\right) =$$

$$= x(t)\dot{x}(t) + y(t)\dot{y}(t) + \cancel{\sqrt{1 - x^2(t) - y^2(t)}} \cdot \frac{-x(t)\dot{x}(t) - y(t)\dot{y}(t)}{\cancel{\sqrt{1 - x^2(t) - y^2(t)}}} = 0$$

($\Leftarrow$) Po predpostavki velja $\forall t \in [a, b] : \vec{r}(t) \cdot \dot{\vec{r}}(t) = 0$. Dokazujemo, da je K na sferi. Naj bo t poljuben.

$$\vec{r}(t) \cdot \dot{\vec{r}}(t) = 0$$

$$(x(t), y(t), z(t)) \cdot (\dot{x}(t), \dot{y}(t), \dot{z}(t)) = x(t)\dot{x}(t) + y(t)\dot{y}(t) + z(t)\dot{z}(t) =$$

$$= \frac{\left(x^2(t) + y^2(t) + z^2(t)\right)'}{2} = 0 \qquad / \int$$

$$\frac{1}{2}\left(x^2(t) + y^2(t) + z^2(t)\right) + C = 0$$

$$x^2(t) + y^2(t) + z^2(t) = -\frac{C}{2} \text{ (konstanta)}$$

Slednje je pogoj za sfero z radijem $\sqrt{-\frac{C}{2}}$, torej $z^2(t) = -\frac{C}{2} - y^2(t) - x^2(t) \Longrightarrow z(t) = \sqrt{-\frac{C}{2} - y^2(t) - x^2(t)}$.

Naloga 3. [1%] Naj bo P rob stožca

$$\sqrt{x^2+y^2} \leq z \leq 1,$$

sestavljen iz plašča in kroga. Izračunaj:

(a) $\iint_P \left(x^2+y^2\right)\, dS$;

Rešitev. Stožec si lahko skiciramo v mislih in parametriziramo plašč C in osnovno ploskev K.

$$\vec{r}\,(z,\varphi) = (z\sin\varphi, z\cos\varphi, z)\,, \quad \vec{k}\,(z,\varphi) = (z\sin\varphi, z\cos\varphi, 1)\,,$$

kjer $z \in [0,1]$ in $\varphi \in [0,2\pi]$ za obe ploskvi. Velja $P = C \cup K$ in $C \cap K$ ima ploščino 0, torej

$$\iint_{P=C\cup K} \left(x^2+y^2\right)\, dS = \iint_C \left(x^2+y^2\right)\, dS + \iint_K \left(x^2+y^2\right)\, dS$$

Izračunajmo parcialne odvode parametrizacije

$$\vec{r_z}\,(z,\varphi) = (\sin\varphi, \cos\varphi, 1)\,, \quad \vec{r_\varphi}\,(z,\varphi) = (z\cos\varphi, -z\sin\varphi, 0)$$

$$\vec{k_z}\,(z,\varphi) = (\sin\varphi, \cos\varphi, 0)\,, \quad \vec{k_\varphi} = (z\cos\varphi, -z\sin\varphi, 0)$$

Uporabimo obrazec za ploskovni integral ($\iint_\pi \rho\, dS = \iint_D \rho\,(\vec{r}\,(u,v)) \cdot \sqrt{(\vec{r_u}\cdot\vec{r_u})\,(\vec{r_v}\cdot\vec{r_v}) - (\vec{r_u}\cdot\vec{r_v})^2}\, dudv$):

$$\iint_C \left(x^2+y^2\right)\, dS = \int_0^{2\pi}\int_0^1 \left(z^2\sin^2\varphi + z^2\cos^2\varphi\right)\cdot\sqrt{EG-F^2}\, dzd\varphi =$$

$$= \int_0^{2\pi}\int_0^1 z^2\sqrt{\left(\sin^2\varphi+\cos^2\varphi+1\right)\left(z^2\cos^2\varphi+z^2\sin^2\varphi\right)-0}\, dzd\varphi =$$

$$= \int_0^{2\pi}\int_0^1 z^2\sqrt{2z^2}\, dzd\varphi = \sqrt{2}\int_0^{2\pi}\int_0^1 z^3\, dzd\varphi = 2\pi\sqrt{2}\int_0^1 z^3 dz =$$

$$= 2\sqrt{2}\pi\left.\frac{z^4}{4}\right|_0^1 = \frac{\pi\sqrt{2}}{2}$$

$$\iint_K \left(x^2+y^2\right)\, dS = \int_0^{2\pi}\int_0^1 \left(z^2\sin^2\varphi + z^2\cos^2\varphi\right)\sqrt{EG-F^2}\, dzd\varphi =$$

$$= \int_0^{2\pi}\int_0^1 z^2\sqrt{1\cdot z^2 - \cancel{(z\sin\varphi\cos\varphi - z\sin\varphi\cos\varphi)^2}}\, dzd\varphi =$$

$$= \int_0^{2\pi}\int_0^1 z^3\, dzd\varphi = 2\pi\int_0^1 z^3\, dz = 2\pi\frac{1}{4} = \frac{\pi}{2}$$

$$\iint_P \left(x^2+y^2\right)\, dS = \frac{\pi}{2} + \sqrt{2}\frac{\pi}{2} = \frac{\pi}{2}\left(1+\sqrt{2}\right)$$

(b) $\iint_P x\, dy\, dz + y\, dz\, dx + z\, dx\, dy$.

Rešitev.

$$\iint_P x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iint_P (x, y, z)\ d\vec{S} =$$

$$= \iint_C (x, y, z)\ d\vec{S} + \iint_K (x, y, z)\ d\vec{S}.$$

$$\pm \iint_C (x, y, z)\ d\vec{S} = \pm \iint_C (x, y, z) \cdot \vec{n}\, dS =$$

$$= \iint_{C'} \left[(z\sin\varphi, z\cos\varphi, z), (\sin\varphi, \cos\varphi, 1), (z\cos\varphi, -z\sin\varphi, 0)\right]\ dzd\varphi =$$

$$= \iint_{C'} (z\sin\varphi, z\cos\varphi, z) \cdot \left(z\sin\varphi, z\cos\varphi, -z\sin^2\varphi - z\cos^2\varphi\right)\ dzd\varphi =$$

$$= \iint_{C'} (z\sin\varphi, z\cos\varphi, z) \cdot (z\sin\varphi, z\cos\varphi, -z)\ dzd\varphi =$$

$$= \iint_{C'} z^2\sin^2\varphi + z^2\cos^2\varphi - z^2\, dzd\varphi = \iint_{C'} \cancel{z^2 - z^2}\, dzd\varphi = 0$$

Nič, ker je polje vzporedno na plašč. Zategadelj

$$\pm \iint_P (x, y, z)\ d\vec{S} = 0 + \iint_K (x, y, z)\ d\vec{S} =$$

$$= \iint_{K'} \left[(z\sin\varphi, z\cos\varphi, 1), (\sin\varphi, \cos\varphi, 0), (z\cos\varphi, -z\sin\varphi, 0)\right]\ dzd\varphi =$$

$$= \iint_{K'} (z\sin\varphi, z\cos\varphi, 1) \cdot \left(0, 0, \cancelto{-z}{-z\sin^2\varphi - z\cos^2\varphi}\right)\ dzd\varphi =$$

$$= \iint_{K'} -z\, dzd\varphi = -\int_0^{2\pi}\int_0^1 z\, dzd\varphi = -2\pi\int_0^1 z\, dz = -2\pi\ \frac{z^2}{2}\Big|_0^1 = -\pi$$

Potemtakem $\iint_P (x, y, z)\ d\vec{S} = \pm\pi$.

Naloga 4. [2%] Naj bo $a > 0$. Valj $x^2 + z^2 = a^2$ izreže iz ploskve, podane z enačbo $ay = xz$, ploskev P. Izračunaj:

(a) njeno površino;

Rešitev. Vpeljimo parametrizacijo krivulje in izračunajmo njena odvoda. Upoštevajmo $y = \frac{xz}{a}$.

$$\vec{r}\,(r,\varphi) = \left(r\cos\varphi, \frac{r^2\sin^2 2\varphi}{2a}, r\sin\varphi\right),\ r \in [0,1]\,, \varphi \in [0, 2\pi]$$

$$\vec{r_r}\,(r,\varphi) = \left(\cos\varphi, \frac{r\sin 2\varphi}{a}, \sin\varphi\right), \quad \vec{r_\varphi}\,(r,\varphi) = \left(-r\sin\varphi, \frac{r\cos 2\varphi}{a}, r\cos\varphi\right)$$

$$P\,(P) = \iint_P 1\,dS = \int_0^{2\pi}\int_0^a \sqrt{EG - F^2}\,drd\varphi$$

$$E = \vec{r_r}\cdot\vec{r_r} = \cos^2\varphi + \frac{r^2\sin^2 2\varphi}{a^2} + \sin^2\varphi = 1 + \frac{r^2\sin^2 2\varphi}{a^2}$$

$$G = r^2\sin^2\varphi + \frac{r^4\cos^2 2\varphi}{a^2} + r^2\cos^2\varphi = r^2 + \frac{r^4\cos^2 2\varphi}{a^2}$$

$$F = \cancel{-r\cos\varphi\sin\varphi} + \frac{r^3\sin 2\varphi\cos 2\varphi}{a^2} + \cancel{r\cos\varphi\sin\varphi} = \frac{r^3\sin 4\varphi}{2a^2}$$

$$F^2 = \frac{r^6\sin^2 4\varphi}{4a^4}$$

$$EG = \left(1 + \frac{r^2\sin^2 2\varphi}{a^2}\right)\left(r^2 + \frac{r^4\cos^2 2\varphi}{a^2}\right) =$$

$$= r^2 + \frac{r^4\cos^2 2\varphi}{a^2} + \frac{r^4\sin^2 2\varphi}{a^2} + \frac{r^6\sin^2 2\varphi\cos^2 2\varphi}{a^4} =$$

$$= r^2 + \frac{r^4\cancel{\left(\cos^2 2\varphi + \sin^2 2\varphi\right)}}{a^2} + \frac{r^6\sin^2 2\varphi\cos^2 2\varphi}{a^4} = r^2 + \frac{r^4}{a^2} + \frac{r^6\sin^2 4\varphi}{4a^4}$$

$$EG - F^2 = r^2 + \frac{r^4}{a^2} + \cancel{\frac{r^6\sin^2 4\varphi}{4a^4} - \frac{r^6\sin^2 4\varphi}{4a^2}}$$

$$\int_0^{2\pi}\int_0^a \sqrt{r^2 + \frac{r^4}{a^2}}\,drd\varphi = 2\pi\int_0^a r\sqrt{1 + \frac{r^2}{a^2}}\,dr =$$

Substituiramo $u = 1 + \frac{r^2}{a^2}$, torej $\sqrt{a^2\,(u-1)} = r$ in $\frac{1}{2\sqrt{a^2(u-1)}}a^2du = dr$ in meji sta sedaj $1 + \frac{0}{a^2} = 1$ in $1 + \frac{a^2}{a^2} = 2$:

$$= 2\pi\int_1^2 \cancel{\sqrt{a^2\,(u-1)}}\sqrt{u}\frac{1}{2\cancel{\sqrt{a^2\,(u-1)}}}a^2du = a^2\pi\int_1^2\sqrt{u}du = a^2\pi\left.\frac{u^{3/2}}{3/2}\right|_1^2 =$$

$$= a^2\pi\frac{4\sqrt{2}}{3} - a^2\pi\frac{2}{3} = a^2\pi\frac{2}{3}\left(2\sqrt{2} - 1\right)$$

(b) dolžino njenega roba (rezultat izrazi s funkcijo beta).

Rešitev. Parametrizirajmo rob z $\vec{r}\,(\varphi) = \left(a\cos\varphi, \frac{a^{\cancel{2}}\sin 2\varphi}{2\cancel{a}}, a\sin\varphi\right)$ za $\varphi \in [0, 2\pi]$, torej $\dot{\vec{r}}\,(\varphi) = (-a\sin\varphi, a\cos 2\varphi, a\cos\varphi)$.

$$\int_{\delta P} 1 d\ell = \int_0^{2\pi} \left|\dot{\vec{r}}\,(\varphi)\right| d\varphi = \int_0^{2\pi} \sqrt{a^2\sin^2\varphi + a^2\cos^2\varphi + a^2\cos^2\varphi} =$$

$$= \int_0^{2\pi} \sqrt{a^2 + a^2\cos^2 2\varphi} d\varphi = \int_0^{2\pi} a\sqrt{1+\cos^2 2\varphi} =$$

$$= a4 \int_0^{\pi/2} \sqrt{1+\cos^2 2\varphi} d\varphi = a4 \int_0^{\pi/2} \sqrt{1+\cos^2\varphi} d\varphi =$$

Substituirajmo $u = \cos\varphi \implies \varphi = \arccos u \implies d\varphi = \frac{-1}{\sqrt{1-u^2}}du$. Meje: $\cos 0 = 1$ in $\cos\frac{\pi}{2} = 0$.

$$= a4 \int_1^0 -\frac{\sqrt{1+u^2}}{\sqrt{1-u^2}} du = a4 \int_0^1 \frac{\sqrt{1+u^2}}{\sqrt{1-u^2}} du = a4 \int_0^1 \frac{1+u^2}{\sqrt{1-u^4}} du =$$

$$= a4 \left(\int_0^1 \frac{1}{\sqrt{1-u^4}} du + \int_0^1 \frac{u^2}{\sqrt{1-u^4}} du\right) =$$

Substituirajmo $t = u^4 \implies u = t^{1/4} \implies du = \frac{1}{4}t^{-3/4}dt$. Meje ostanejo.

$$= a\cancel{4} \left(\frac{1}{\cancel{4}} \int_0^1 (1-t)^{-1/2}\, t^{-3/4} dt + \frac{1}{\cancel{4}} \int_0^1 (1-t)^{-1/2} \cancelto{t^{-1/4}}{\sqrt{t}t^{-3/4}} dt\right) =$$

$$= a\left(B\left(\frac{1}{2}, \frac{1}{4}\right) + B\left(\frac{1}{2}, \frac{3}{4}\right)\right)$$

Naloga 5. [2%] Kompleksna funkcija f na območju $D = \{z \in \mathbb{C} \mid 0 < \arg z < \pi/2\}$ je podana s predpisom

$$f(x+iy) = \frac{x^5(x-2iy)\left((\ln x - \ln y)^2 + i\right)}{(x^2+4y^2)(x^4+y^4)}.$$

Izračunaj integral

$$\int_K f(z)\,dz,$$

kjer je krivulja K:

(a) daljica od izhodišča do poljubne točke na premici $y = x$;

Rešitev. D predstavlja pri kvadrant. Točke na tej daljici so točke oblike $t(1+i)$ za $t \in [0,a]$. Parametrizacija: $r(t) = t(1+i)$, $\dot{r}(t) = 1+i$. Vstavimo $x = t$ in $y = t$ v f.

$$\int_K f(z)\,dz = \int_0^a \frac{t^5(t-2it)\left(\cancel{(\ln t - \ln t)^2} + i\right)}{(t^2+4t^2)(t^4+t^4)}(1+i)\,dt =$$

$$= \int_0^a \frac{t^6(1-2i)\,i\,(1+i)}{5t^2 2t^4}dt = \int_0^a \frac{\cancel{t^6}(i+2)(1+i)}{10\cancel{t^6}}dt = \frac{1}{10}\int_0^a i - 1 + 2 + 2i\,dt =$$

$$= \frac{1}{10}\int_0^a 3i+1\,dt = \frac{a}{10}(3i+1)$$

(b) parabola $y = x^2$ za $x > 0$.

Rešitev. Zopet D predstavlja prvi kvadrant. Točke na tej paraboli so točke oblike $t + t^2 i$ za $t \in [0,\infty)$. Parametrizacija: $r(t) = t(1+ti)$, $\dot{r}(t) = 1 + 2ti$. Vstavimo $x = t$ in $y = t^2$ v f.

$$\int_K f(z)\,dz = \int_0^\infty \frac{t^5\left(t-2it^2\right)\left(\left(\ln t - \ln t^2\right)^2 + i\right)}{(t^2+4t^4)(t^4+t^8)}(1+2ti)\,dt =$$

$$= \int_0^\infty \frac{\cancel{t^6}(1-2it)\left((-\ln t)^2 + i\right)}{\cancel{t^2}(1+4t^2)\cancel{t^4}(1+t^4)}(1+2ti)\,dt = \int_0^\infty \frac{\cancel{(1+4t^2)}\left((-\ln t)^2+i\right)}{\cancel{(1+4t^2)}(1+t^4)}dt =$$

$$= i\int_0^\infty \frac{1}{1+t^4}dt + \int_0^\infty \frac{\ln^2 t}{1+t^4}dt =$$

Substituiramo $u = t^4$, $t = \sqrt[4]{u}$, $dt = \frac{u^{-3/4}}{4}$.

$$= \frac{i}{4}\int_0^\infty \frac{u^{-3/4}}{1+u}du + \int_0^\infty \frac{\ln^2 t}{1+t^4}dt =$$

$p-1=-3/4 \Longrightarrow p=1/4,\ p+q=1 \Longrightarrow q=3/4$

$$=\frac{i}{4}B\left(\frac{1}{4},\frac{3}{4}\right)+\int_0^\infty \frac{\ln^2 t}{1+t^4}dt=\cdots$$

Sedaj si oglejmo $B\left(p,1-p\right)=\frac{\pi}{\sin \pi p}$, natančneje odvode funkcije $p \mapsto B\left(p,1-p\right)$. Naj bo $b\left(x\right):=B\left(x,1-x\right)$.

$$b\left(x\right)=\int_0^\infty \frac{u^{x-1}}{\left(1+u\right)^1}du=\frac{\pi}{\sin \pi x}$$

$$b'\left(x\right)=\int_0^\infty \frac{u^{x-1}\ln u}{1+u}du=\frac{-\pi^2\cos \pi x}{\sin^2 \pi x}$$

$$b''\left(x\right)=\int_0^\infty \frac{u^{x-1}\ln^2 u}{1+u}du=-\pi^2\left(\frac{\cos \pi x}{\sin^2 \pi x}\right)'=$$

$$=-\pi^2\frac{-\pi \sin \pi x \sin^2 \pi x-\cos \pi x\left(\sin^2 \pi x\right)'}{\sin^4 \pi x}=$$

$$=\pi^3\frac{\sin^3 \pi x+\cos\left(\pi x\right)2\sin\left(\pi x\right)\cos\left(\pi x\right)}{\sin^4 \pi x}=\pi^3\frac{\sin^3 \pi x+2\cos^2 \pi x \sin \pi x}{\sin^4 \pi x}$$

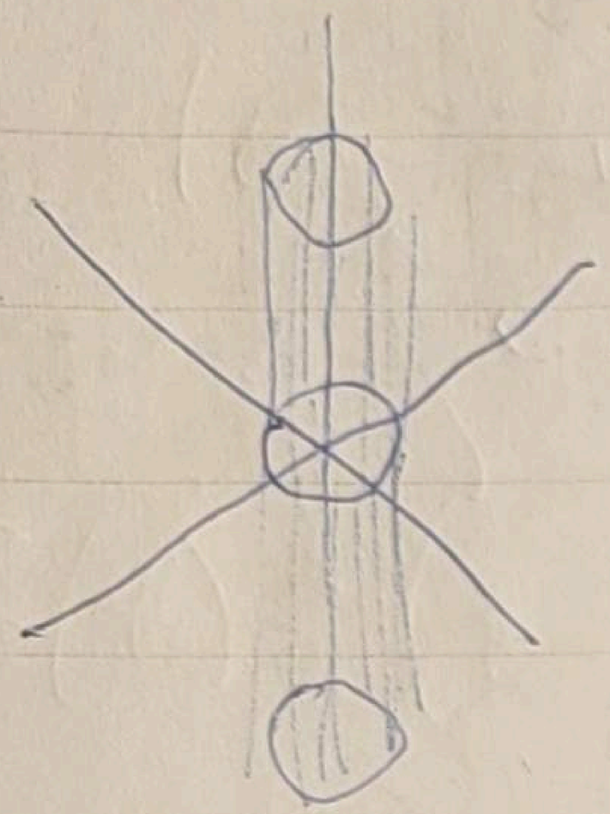
s tem znanjem v integral substituirajmo $u=t^4 \Longrightarrow \sqrt[4]{u}=t \Longrightarrow \frac{1}{4}u^{-3/4}du=dt$:

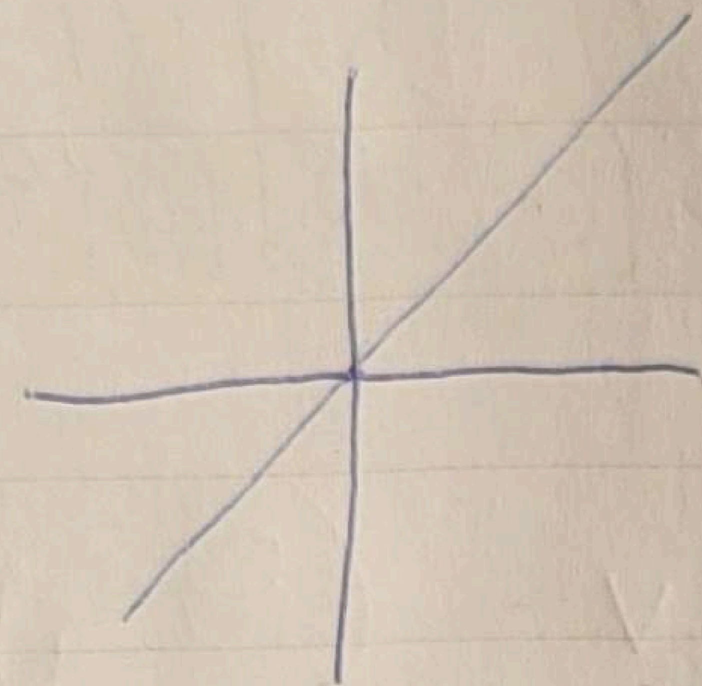
$$\cdots=\int_0^\infty \frac{\ln^2\left(u^{1/4}\right)u^{-3/4}}{1+u}\cdot\frac{1}{4}du+\frac{i}{4}b\left(\frac{1}{4}\right)=\frac{1}{64}\int_0^\infty \frac{u^{-3/4}\ln^2 u}{1+u}du+\frac{i\pi\sqrt{2}}{4}=$$

$$=\frac{1}{64}b''\left(\frac{1}{4}\right)+\frac{\pi\sqrt{2}}{4}i=\frac{\pi^3\left(\frac{1}{\sqrt{2}^3}+\frac{2}{\sqrt{2}^2\sqrt{2}}\right)}{64\cdot\frac{1}{\sqrt{2}^4}}+\frac{\pi\sqrt{2}}{4}i=\frac{\pi^3\left(\frac{1}{2\sqrt{2}}+\frac{2}{2\sqrt{2}}\right)}{64\cdot\frac{1}{4}}+\frac{\pi\sqrt{2}}{4}i=$$

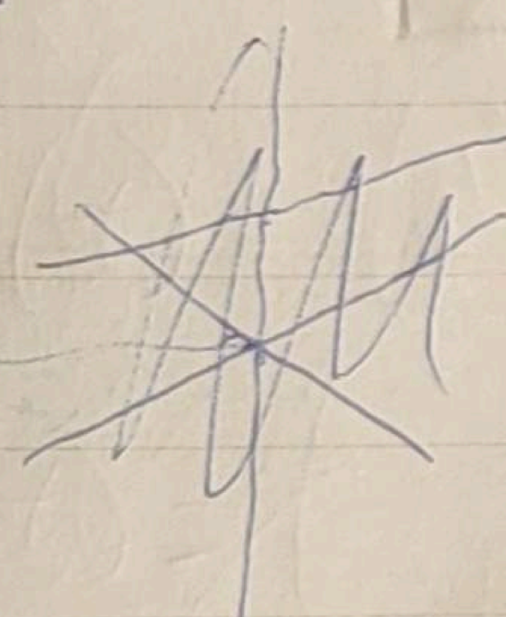
$$=\frac{\pi^3 3}{16\sqrt{2}2}+\frac{\pi\sqrt{2}}{4}i=\frac{\pi^3 3}{32\sqrt{2}}+\frac{\pi\sqrt{2}}{4}i$$

1. $x^2+y^2=1$ $\quad x\tan z=y$; $z=\operatorname{atan}\frac{y}{x}$





ogledmo si $z=\frac{y}{x}$



$\dot{\vec{r}}(\varphi)=(-\sin\varphi,\cos\varphi,1)$

parametrizacija

$x=\cancel{r\cos\varphi}\ r\cos\varphi$

$y=\cancel{r\cos\varphi}\ r\sin\varphi$

$z=\operatorname{atan} y/x=$

$=\operatorname{atan}\frac{r\sin\varphi}{r\cos\varphi}=$

$=\operatorname{atan}\tan\varphi=\varphi$

$r=1$

$\vec{r}(t)=(\cos\varphi,\sin\varphi,\varphi)\quad t\in[0,2\pi]$

$\int_C y\,dx+x\,dy+z\,dz=$

$\int_C (y,x,z)\,d\vec{s}=\int_0^{2\pi}\cancel{(\ldots\cos\varphi,\sin\varphi,\varphi)}\cdot$

$\vec{V}:\mathbb{R}^3\to\mathbb{R}^3$ s predpisom

$(x,y,z)\mapsto(y,x,z)$

$\int_0^{2\pi}\vec{V}(\vec{r}(\varphi))\cdot\dot{\vec{r}}(\varphi)\,d\varphi=$

$=\int_0^{2\pi}(\sin\varphi,\cos\varphi,\varphi)\cdot(-\sin\varphi,\cos\varphi,1)\,d\varphi=$

$=\int_0^{2\pi}-\sin^2\varphi+\cos^2\varphi+\varphi\,d\varphi=$

$=-\int_0^{2\pi}\sin^2\varphi+\int_0^{2\pi}\cos^2\varphi+\frac{\varphi^2}{2}\Big|_0^{2\pi}=$

$=\int_0^{2\pi}\cos^2\varphi-\int_0^{2\pi}\sin^2\varphi+\frac{4\pi^2}{2}=$

$=\int_0^{2\pi}(\cos^2\varphi-\sin^2\varphi)\,d\varphi+2\pi^2=\int_0^{2\pi}\cos(2\varphi)\,d\varphi+2\pi^2=$

$=0+2\pi^2=2\pi^2$

2. ($\Rightarrow$)

$\vec{r}(t) \cdot \dot{\vec{r}}(t) \overset{?}{=} 0$

za sferično parametrizacijo krivulje BSŠ krogla radija 1

$\cancel{\vec{r}(t) = (1, \varphi, \theta)}$

sferičnih koordinatah

$\cancel{\vec{r}(t) = (}$

$\cancel{\vec{r}(t) = (r\cos\varphi\cos\theta, r\sin\varphi\cos\theta, z\sin\theta)}$

$\cancel{\dot{\vec{r}}(t) = (}$ $\vec{r}(t) = (x(t), y(t), \sqrt{1-x^2(t)-y^2(t)})$

$\dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t), (\sqrt{1-x^2(t)-y^2(t)})')$

$\vec{r}(t) \cdot \dot{\vec{r}}(t) = x(t)\dot{x}(t) + y(t)\dot{y}(t) +$

$+ \sqrt{1-x^2(t)-y^2(t)} \cdot \frac{1}{2\sqrt{1-x^2(t)-y^2(t)}} \cdot (1-x^2(t)-y^2(t))' =$

$= x(t)\dot{x}(t) + y(t)\dot{y}(t) + \frac{1}{2} \cdot (-2x(t)\dot{x}(t) - 2y(t)\dot{y}(t)) =$

$= x(t)\dot{x}(t) + y(t)\dot{y}(t) - (x(t)\dot{x}(t) + y(t)\dot{y}(t)) = 0$

($\Leftarrow$)

$\vec{r}(t) \cdot \dot{\vec{r}}(t) = 0$

$\vec{r}(t) = (x(t), y(t), z(t))$

$\dot{\vec{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$

$0 = (x\dot{x})(t) + (y\dot{y})(t) + (z\dot{z})(t) =$

$= x(t)\dot{x}(t) + y(t)\dot{y}(t) + z(t)\dot{z}(t)$

$\Rightarrow z(t)\dot{z}(t) = -x(t)\dot{x}(t) - y(t)\dot{y}(t)$

$zz' = -xx' - yy'$

$\frac{1}{2}(x^2(t) + y^2(t) + z^2(t))' = 0 \quad /\int$

$\frac{1}{2}(x^2(t) + y^2(t) + z^2(t)) + C = 0$

$x^2(t) + y^2(t) + z^2(t) = \boxed{-\frac{1}{2}C}$

konstanta

pogoj za sfero, ata

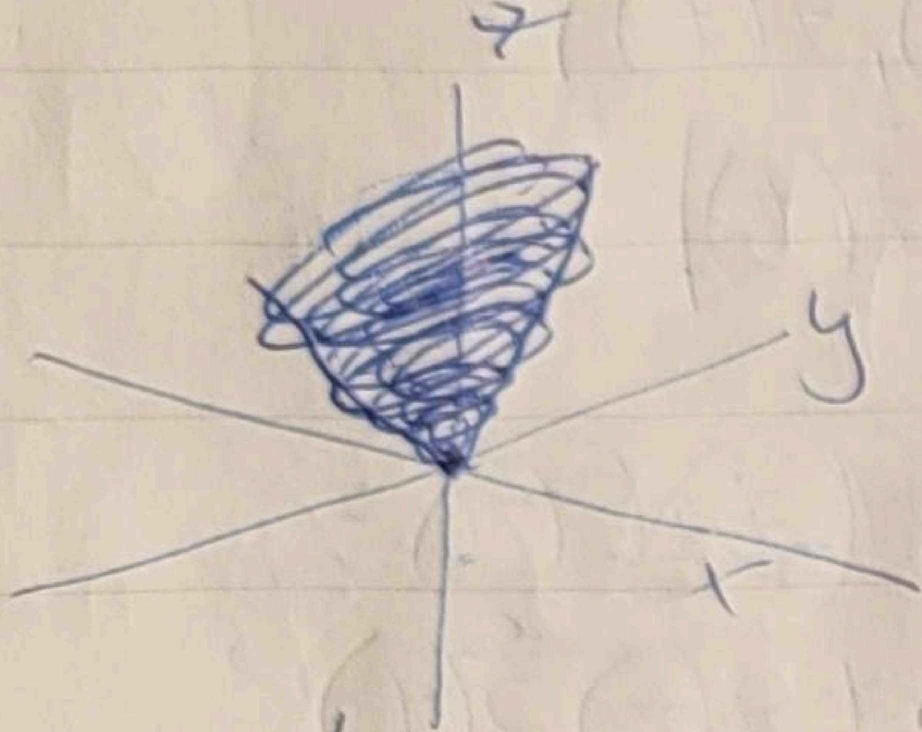
$z^2(t) = -\frac{1}{2}C - y^2(t) - x^2(t)$

radija $\sqrt{-\frac{1}{2}C}$

$z(t) = \sqrt{-1/2C - y^2(t) - x^2(t)}$

3. ~~Naj bo~~

stožec $\sqrt{x^2+y^2} \le z \le 1$



parametrizacija:

plašč C

$\vec{r}(z,\varphi) = (z\sin\varphi, z\cos\varphi, z)$

krog K

$\vec{k}(z,\varphi) = (z\sin\varphi, z\cos\varphi, 1)$

$z \in [0,1]$
$\varphi \in [0,2\pi]$ pri obeh

$P = C \cup K$

(a) $\iint_P (x^2+y^2)\,dS = \iint_C (x^2+y^2)\,dS + \iint_K (x^2+y^2)\,dS$

$\vec{r}_z(z,\varphi) = (\sin\varphi, \cos\varphi, 1)$

$\vec{r}_\varphi(z,\varphi) = (z\cos\varphi, -z\sin\varphi, 0)$

$\vec{k}_z(z,\varphi) = (\sin\varphi, \cos\varphi, 0)$

$\vec{k}_\varphi(z,\varphi) = (z\cos\varphi, -z\sin\varphi, 0)$

$\iint_P f\,dS = \iint_D f(\vec{r}(u,v))\cdot\sqrt{EG-F^2}\,du\,dv$

$E = \vec{r}_u\cdot\vec{r}_u \quad G = \vec{r}_v\cdot\vec{r}_v$

$F = \vec{r}_u\cdot\vec{r}_v$

$\iint_C (x^2+y^2)\,dS = \iint_{C'} (z^2\sin^2\varphi + z^2\cos^2\varphi)\cdot\sqrt{EG-F^2}\,dz\,d\varphi$

$= \iint_{C'} (z^2\sin^2\varphi + z^2\cos^2\varphi)\sqrt{(\sin^2\varphi+\cos^2\varphi+1)(z^2\cos^2\varphi+z^2\sin^2\varphi) - (\cancel{z\sin\varphi\cos\varphi} - \cancel{z\sin\varphi\cos\varphi})^2}\,dz\,d\varphi =$

$= \iint_{C'} z^2\sqrt{2z^2}\,dz\,d\varphi =$

$= \sqrt{2}\iint_{C'} z^3\,dz\,d\varphi =$

$= \sqrt{2}\, 2\pi \int_0^1 z^3\,dz = \sqrt{2}\, 2\pi \frac{z^4}{4}\Big|_0^1 =$

$= \sqrt{2}\cdot 2\pi\cdot\frac{1}{4} = \frac{\sqrt{2}\pi}{2}$

$$\iint_{L} (x^2+y^2)\, dS = \iint_{L'} (z^2\sin^2\varphi + z^2\cos^2\varphi)\sqrt{EG-F^2}\, dz\, d\varphi$$

$$= \iint_{L'} z^2 \sqrt{1\cdot z^2 - (z\sin\varphi\cos\varphi - z\sin\varphi\cos\varphi)^2}\, dz\, d\varphi =$$

$$= \iint_{L'} z^2 \sqrt{z^2}\, dz\, d\varphi = \iint_{L'} z^3\, dz\, d\varphi =$$

$$= 2\pi \int_0^1 z^3\, dz = 2\pi \frac{1}{4} = \pi/2$$

$$\iint_P (x^2+y^2)\, dS = \frac{\pi}{2} + \sqrt{2}\frac{\pi}{2} = \frac{\pi}{2}(1+\sqrt{2})$$

(b) $\iint_P x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

$$\iint_P (x,y,z)\, d\vec{S} = \iint_C (x,y,z)\, d\vec{S} + \iint_L (x,y,z)\, d\vec{S}$$

$$\iint_C (x,y,z)\, d\vec{S} = \iint_C (x,y,z)\cdot\vec{n}\, dS =$$

$$= \pm\iint_{C'} [(z\sin\varphi, z\cos\varphi, z), (\sin\varphi, \cos\varphi, 1), (z\cos\varphi, -z\sin\varphi, 0)]\, dz\, d\varphi$$

$$= +\iint_{C'} (z\sin\varphi, z\cos\varphi, z)\cdot(z\sin\varphi, z\cos\varphi, -z\sin^2\varphi - z\cos^2\varphi)\, dz\, d\varphi =$$

$$= \iint_{C'} (z\sin\varphi, z\cos\varphi, z)(z\sin\varphi, z\cos\varphi, -z)\, dz\, d\varphi =$$

$$= \iint_{C'} z^2\sin^2\varphi + z^2\cos^2\varphi - z^2\, dz\, d\varphi =$$

$$= \iint_{C'} z^2 - z^2\, dz\, d\varphi = \iint_{C'} 0\, dz\, d\varphi = 0$$

polje vzporedno na ~~vab~~ plašč

$$\pm\iint_P (x,y,z)\, d\vec{S} = \iint_L (x,y,z)\, d\vec{S} =$$

$$= \iint_{L'} [(z\sin\varphi, z\cos\varphi, 1), (\sin\varphi, \cos\varphi, 0), (z\cos\varphi, -z\sin\varphi, 0)]\, dz\, d\varphi =$$

$$= \iint_{L'} (z\sin\varphi, z\cos\varphi, 1)\cdot(0, 0, \underbrace{-z\sin^2\varphi - z\cos^2\varphi}_{-z})\, dz\, d\varphi$$

$$= \iint_{L'} -z\, dz\, d\varphi = -\iint_{L'} z\, dz\, d\varphi = -2\pi\int_0^1 z\, dz =$$

$$= -2\pi \frac{z^2}{2}\Big|_0^1 = -\pi$$

$$\iint_P (x,y,z)\, d\vec{S} = \pm\pi$$

4.a) $x^2+z^2=a^2 \quad y=\frac{xz}{a}$

$ay=xz$

$\vec{r}(r,\varphi)=\left(r\cos\varphi, \frac{r^2\sin 2\varphi}{2a}, r\sin\varphi\right)$

$\iint_P 1\,dS = \int_0^{2\pi}\int_0^a \sqrt{EG-F^2}\,dr\,d\varphi$

$\vec{r}_r(r,\varphi)=\left(\cos\varphi, \frac{r\sin 2\varphi}{a}, \sin\varphi\right)$

$\vec{r}_\varphi(r,\varphi)=\left(-r\sin\varphi, \frac{r^2\cos 2\varphi}{a}, r\cos\varphi\right)$

$E=\left(\cos^2\varphi+\frac{r^2\sin^2 2\varphi}{a^2}+\sin^2\varphi\right)=1+\frac{r^2\sin^2 2\varphi}{a^2}$

$G=\left(r^2\sin^2\varphi+\frac{r^4\cos^2 2\varphi}{a^2}+r^2\cos^2\varphi\right)=r^2+\frac{r^4\cos^2 2\varphi}{a^2}$

$F=\cancel{-r\cos\varphi\sin\varphi}+\frac{r^3\sin 2\varphi\cos 2\varphi}{a^2}+r\cos\varphi\sin\varphi=$

$=\frac{r^3\sin 4\varphi}{2a^2}$

$F^2=\frac{r^6\sin^2 4\varphi}{4a^4}$

$EG=\left(1+\frac{r^2\sin^2 2\varphi}{a^2}\right)\left(r^2+\frac{r^4\cos^2 2\varphi}{a^2}\right)=$

$=r^2+\frac{r^4\cos^2 2\varphi}{a^2}+\frac{r^4\sin^2 2\varphi}{a^2}+\frac{r^6\sin^2 2\varphi\cos^2 2\varphi}{a^4}=$

$=r^2+\frac{r^4\cancel{(\cos^2 2\varphi+\sin^2 2\varphi)}}{a^2}+\cancel{\frac{r^6\sin^2 2\varphi\sin^2 4\varphi}{2a^4}}$

$+\left(\frac{r^6\sin^2 2\varphi\cos^2 2\varphi}{a^4}=\frac{r^6\sin^2 4\varphi}{4a^4}\right)$

$$EG-F^2 = r^2 + \frac{r^4}{a^2} + \cancel{\frac{r^2\sin^2 2\varphi}{4a^2}} - \cancel{\frac{r^2\sin^2 2\varphi}{4a^4}} =$$

$$= r^2 + \frac{r^4}{a^2}$$

$$\sqrt{EG-F^2} = \sqrt{r^2\left(1+\frac{r^2}{a^2}\right)} = r\sqrt{1+r^2/a^2}$$

$$\int_0^{2\pi}\int_0^a r\sqrt{1+r^2/a^2}\,dr\,d\varphi =$$

$$= 2\pi\int_0^a r\sqrt{1+r^2/a^2}\,dr =$$

$$u = 1 + r^2/a^2$$

$$\sqrt{a^2(u-1)} = r$$

$$\frac{1}{2\sqrt{a^2(u-1)}}a^2\,du = dr$$

$$= 2\pi\int_1^2 \cancel{\sqrt{a^2(u-1)}}\sqrt{u}\,\frac{1}{2\cancel{\sqrt{a^2(u-1)}}}a^2\,du =$$

$$= \pi a^2\int_1^2\sqrt{u}\,du = \pi a^2\frac{u^{3/2}}{3/2}\Big|_1^2 =$$

$$\cancel{= \pi a^2\frac{2^{3/2}\cdot 2}{3} = \pi a^2\frac{\sqrt{8}\cdot 2}{3}}$$

$$\cancel{\frac{4}{3}\cdot\sqrt{2}\cdot\pi\cdot a^2}$$

$$= \pi a^2\frac{2\sqrt{2}\cdot 2}{3} - \pi a^2\frac{1\cdot 2}{3} =$$

$$= \pi a^2\frac{2}{3}\left(2\sqrt{2}-1\right)$$

b.) parametriziramo rob:

$$\vec{r}(\varphi) = \left(a\cos\varphi, \frac{a^2\sin 2\varphi}{2a}, a\sin\varphi\right) =$$

$$= \left(a\cos\varphi, \frac{a\sin 2\varphi}{2}, a\sin\varphi\right)$$

$$\dot{\vec{r}}(\varphi) = \left(-a\sin\varphi, a\cos 2\varphi, a\cos\varphi\right)$$

$$\int_{\partial P} 1\,dl = \int_0^{2\pi}|\dot{\vec{r}}(\varphi)|\,d\varphi =$$

$$|\dot{\vec{r}}(\varphi)| = \sqrt{a^2\sin^2\varphi + a^2\cos^2 2\varphi + a^2\cos^2\varphi} =$$

$$= \sqrt{a^2 + a^2\cos^2 2\varphi} = a\sqrt{1+\cos^2 2\varphi}$$

$$a\int_0^{2\pi}\sqrt{1+\cos^2 2\varphi}\,d\varphi = a4\int_0^{\pi/2}\sqrt{1+\cos^2 2\varphi}\,d\varphi =$$

$$= a4\int_0^{\pi/2}\sqrt{1+\underbrace{\cos^2\varphi}_{1-\sin^2 x}}\,d\varphi =$$

$$= a4\int_0^{\pi/2}\sqrt{2-\sin^2\varphi}\,d\varphi = a4\int_0^{1}\frac{\sqrt{2-u^2}}{\sqrt{1-u^2}}\,du =$$

$u=\sin\varphi$ $\quad \sin(0)=0$

$\varphi=\operatorname{asin} u$ $\quad \sin(\pi/2)=1$

$d\varphi=\frac{1}{\sqrt{1-u^2}}du$

$$= a4\int_0^1\frac{\sqrt{2-u^2}}{\sqrt{1-u^2}}\,du =$$

$$= a4\int_0^1\frac{2-u^2}{\sqrt{(1-u^2)(2-u^2)}}\,du =$$

$u=\cos\varphi$ $\quad \cos(0)=1$

$\varphi=\operatorname{acos} u$ $\quad \cos(\pi/2)=0$

$d\varphi=-\frac{1}{\sqrt{1-u^2}}du$

$$= a4\int_1^0 -\frac{\sqrt{1+u^2}}{\sqrt{1-u^2}}\,du = a4\int_0^1\frac{\sqrt{1+u^2}}{\sqrt{1-u^2}}\,du =$$

$$= a4\int_0^1\frac{1+u^2}{\sqrt{1-u^4}}\,du = a4\left(\int_0^1\frac{1}{\sqrt{1-u^4}}du+\int_0^1\frac{u^2}{\sqrt{1-u^4}}du\right)=$$

$t=u^4,\ u=t^{1/4}$

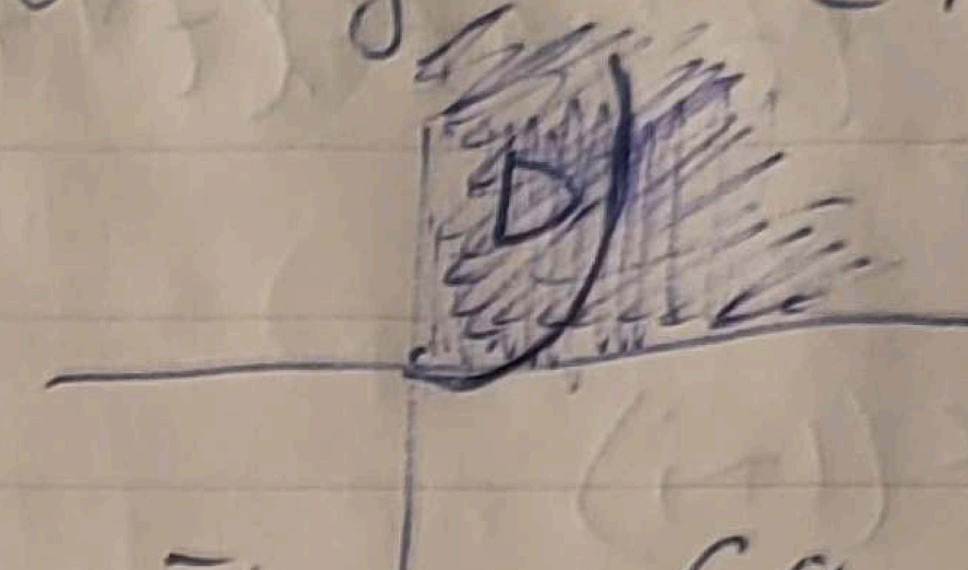
$du=\frac{1}{4}t^{-3/4}$

$$= a\left(\int_0^1(1-t)^{-1/2}t^{-3/4}\,dt+\frac{1}{4}\int_0^1(1-t)^{-1/2}\underbrace{\sqrt{t}\,t^{-3/4}}_{t^{-1/4}}\,dt\right)=$$

$$= a\left(B(1/2,1/4)+B(1/2,3/4)\right)$$

5.

$D=\{z\in\mathbb{C} : 0<\arg z<\pi/2\}$



$$f(x+iy)=\frac{x^5(x-2iy)\left((\ln x-\ln y)^2+i\right)}{(x^2+4y^2)(x^4+y^4)}$$

a.) $$\int_C f(z)\,dz=\int_0^a\frac{t^5(t-2it)\left((\ln t-\ln t)^2+i\right)}{(t^2+4t^2)(t^4+t^4)}\,dt=\int_0^a\frac{t^5(t-2ti)i}{5t^2\cdot 2t^4}\,dt=\int_0^a\frac{t(1-2i)i\,t^4}{t^2(5+2t^2)}\,dt=$$

$$= i\int_0^a\frac{1-2i}{5+2t^2}\,dt = i\int_0^a\frac{1}{5+2t^2}\,dt+\int_0^a\frac{2}{5+2t^2}\,dt$$

$$-\int_0^a$$

a) $\cancel{\int_t f(z)dz = \int_0^a \frac{t^5(t-2it)((\ln t - \ln t)^2+i)}{(t^2+4t^2)(t^4+t^4)}dt}$

- $r(t) = t(1+i)$

$\dot{r}(t) = 1+i$

$$\int_t f(z)dz = \int_0^a \frac{t^5(t-2it)((\ln t - \ln t)^2+i)(1+i)}{(t^2+4t^2)(t^4+t^4)}dt$$

$$= \int_0^a \frac{\cancel{t^6}(1-2i)\,i(1+i)}{\cancel{10t^6}}dt = \int_0^a \frac{(1-2i)i(1+i)}{10}dt =$$

$$= \frac{1}{10}\int_0^a (i+2)(1+i)\,dt =$$

$$= \frac{1}{10}\int_0^a i-1+2+2i\,dt = \frac{1}{10}\int_0^a 3i+1\,dt =$$

$$= \frac{a}{10}3i + \frac{a}{10} = \frac{a}{10}(3i+1)$$

b.) $r(t) = t + t^2 i = t(1+ti)$

$\dot{r}(t) = 1 + 2ti$

$y = t^2 \quad x = t$

$$\int_t f(z)dz = \int_0^\infty \frac{t^5(t-2it^2)((\ln t - \ln t^2)^2+i)(1+2ti)}{(t^2+4t^4)(t^4+t^8)}dt =$$

$$= \int_0^\infty \frac{\cancel{t^6}(1-2ti)((\ln t)^2+i)(1+2ti)}{\cancel{t^2}(1+4t^2)\cancel{t^4}(1+t^4)}dt =$$

$$= \int_0^\infty \frac{(1-2ti)(i+(\ln t)^2)(1+2ti)}{(1+4t^2)(1+t^4)}dt =$$

$$= \int_0^\infty \frac{\cancel{(1+4t^2)}(i+(\ln t)^2)}{\cancel{(1+4t^2)}(1+t^4)}dt = \int_0^\infty \frac{i+(\ln t)^2}{1+t^4}dt =$$

$$= i\int_0^\infty \frac{1}{1+t^4}dt + \int_0^\infty \frac{(\ln t)^2}{1+t^4}dt =$$

$u = t^4 \quad du = 4t^3dt \quad t = \sqrt[4]{u} \quad dt = \frac{1}{4}u^{-3/4}du$

$p-1 = -3/4$

$p = 1/4$

$p+q = 1$

$q = 3/4$

$$= \int_0^\infty \frac{(\ln t)^2}{1+t^4}dt + i\frac{1}{4}\int_0^\infty \frac{u^{-3/4}}{1+u}du =$$

$$= \int_0^\infty \frac{(\ln t)^2}{1+t^4}dt + i\frac{1}{4}B\left(\frac{1}{4},\frac{3}{4}\right) =$$

$$= \int_0^\infty \frac{(\ln t)^2}{1+t^4}\, dt + i/4\, B(1/4, 3/4) = \dots$$

$u = t^4 \qquad \sqrt[4]{u} = t \qquad \frac{1}{4} \cdot u^{-3/4} du = dt$

oglejmo si $B(p, 1-p) = \frac{\pi}{\sin \pi p}$,

natančneje odvode $p \mapsto B(p, 1-p)$:

$b(x) := B(x, 1-x)$.

$$b(x) = \int_0^\infty \frac{u^{x-1}}{(1+u)^1} du = \frac{\pi}{\sin \pi x}$$

$$b'(x) = \int_0^\infty \frac{u^{x-1} \ln u}{(1+u)} du = \frac{-\pi^2 \cos \pi x}{\sin^2 \pi x}$$

$$b''(x) = \int_0^\infty \frac{u^{x-1} \ln^2 u}{1+u} du =$$

$$= -\pi^2 \left(\frac{\cos \pi x}{\sin^2 \pi x}\right)' = -\pi^2 \frac{-\pi \sin \pi x \sin^2 \pi x - \cos \pi x (\sin^2 \pi x)'}{\sin^4 \pi x}$$

$$= +\pi^3 \frac{\sin^3 \pi x + \cos(\pi x) 2 \sin(\pi x) \cos(\pi x)}{\sin^4(\pi x)} = \pi^3 \frac{\sin^3 \pi x + 2\cos^2 \pi x \sin \pi x}{\sin^4 \pi x}$$

$$\dots = \int_0^\infty \frac{\ln(u^{1/4})^2 u^{-3/4}}{1+u} \cdot \frac{1}{4} du + \frac{i}{4} b\left(\frac{1}{4}\right) =$$

$$= \frac{1}{64} \int_0^\infty \frac{\ln^2 u\, u^{-3/4}}{1+u} du + \frac{i}{4} b\left(\frac{1}{4}\right) =$$

$$= \frac{1}{64} b''(1/4) + \frac{i}{4} b\left(\frac{1}{4}\right) \cancel{+ \frac{\pi}{\sin\frac{\pi}{4}}\left(\frac{1}{64} + \frac{i}{4}\right)} =$$

$$\cancel{= \frac{\pi}{\sqrt{2}}\left(\frac{1}{64} + \frac{i}{4}\right)}$$

$$\cancel{= \frac{\pi^3 \sqrt{2}^3 + 2\sqrt{2}^2\sqrt{2}}{64 \sqrt{2}^4} + \frac{1}{4} \cdot \frac{\pi}{\sqrt{2}} i}$$

$$\cancel{= \frac{\pi^3 2\sqrt{2} + 4\sqrt{2}}{256} + \frac{\pi}{4\sqrt{2}} i}$$

$$\cancel{= \frac{\sqrt{2}(\pi^3 + 2)}{128} + \frac{\pi}{4\sqrt{2}} i}$$

$$= \frac{\pi^3 \left(\frac{1}{\sqrt{2}^3} + \frac{2}{\sqrt{2}^2 \sqrt{2}}\right)}{64 \cdot \frac{1}{\sqrt{2}^4}} + \frac{\pi \sqrt{2}}{4} i =$$

$$= \frac{\pi^3 \left(\frac{1}{2\sqrt{2}} + \frac{2}{2\sqrt{2}}\right)}{64 \cdot \frac{1}{4}} + \frac{\pi\sqrt{2}}{4} i = \frac{\pi^3 \cdot 3}{16 \sqrt{2}\, 2} + \frac{\pi\sqrt{2}}{4} i =$$

$$= \frac{\pi^3 \cdot 3}{32\sqrt{2}} + \frac{\pi\sqrt{2}}{4} i$$